

# GEOMETRIC DUALITY THEORY OF CONES IN DUAL PAIRS OF VECTOR SPACES

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**ABSTRACT.** This paper will generalize what may be termed the “geometric duality theory” of real pre-ordered Banach spaces which relates geometric properties of a closed cone in a real Banach space, to geometric properties of the dual cone in the dual Banach space. We show that geometric duality theory is not restricted to real pre-ordered Banach spaces, as is done classically, but can be extended to real Banach spaces endowed with arbitrary collections of closed cones.

We define geometric notions of normality, conormality, additivity and coadditivity for members of dual pairs of real vector spaces as certain possible interactions between two cones and two convex sets containing zero. We show that, thus defined, these notions are dual to each other under certain conditions, i.e., for a dual pair of real vector spaces  $(Y, Z)$ , the space  $Y$  is normal (additive) if and only if its dual  $Z$  is conormal (coadditive) and vice versa. These results are set up in a manner so as to provide a framework to prove results in the geometric duality theory of cones in real Banach spaces. As an example of using this framework, we generalize classical duality results for real Banach spaces pre-ordered by a single closed cone, to real Banach spaces endowed with an arbitrary collections of closed cones.

As an application, we analyze some of the geometric properties of naturally occurring cones in  $C^*$ -algebras and their duals.

## 1. INTRODUCTION

The goal of this paper is to provide a general framework for proving results in what may be termed the “geometric duality theory” of cones in real Banach spaces. We will prove The General Duality Theorems (Theorems 3.4 and 3.5) for dual pairs of real vector spaces and, as an application, we generalize classical results for real pre-ordered Banach spaces (cf. Theorems 1.2 and 1.3) to the context of real Banach spaces endowed with arbitrary collections of closed cones (cf. Corollaries 4.7 and 4.9).

We begin with some motivating historical remarks:

Andô’s Theorem [2, Lemma 1], a fundamental result in the geometric theory of real pre-ordered Banach spaces, states:

**Theorem 1.1.** *Let  $X$  be a real Banach space, pre-ordered by a closed cone  $X_+$  (in the sense that  $x \leq y$  means  $y \in x + X_+$ ). If the cone  $X_+$  generates  $X$ , i.e.,  $X = X_+ - X_+$ , then there exists a constant  $\alpha \geq 1$ , such that every  $x \in X$  can be written as  $x = a - b$  with  $a, b \in X_+$  and  $\max\{\|a\|, \|b\|\} \leq \alpha \|x\|$ .*

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The geometric property given by the conclusion of Andô's Theorem is what is termed a 'conormality property'<sup>1</sup> (cf. property (2)(a) in Theorem 1.2 for another example).

It has long been known that there exists a geometric duality theory for such relationships between norms and closed cones in real pre-ordered Banach spaces. Loosely speaking, there is a dual notion to conormality, called 'normality'<sup>2</sup> (cf. property (1)(a) in Theorem 1.2 for a specific example of a normality property). Normality is 'dual' to conormality in the sense that a real pre-ordered Banach space has a conormality property if and only if its dual has a corresponding normality property, and vice versa. Many such dual pairs of normality- and conormality properties have been discovered for real pre-ordered Banach spaces. A fairly complete inventory of normality and conormality properties and their relationships as dual properties is given with full references in [16, Definition 3.1, Theorem 3.7].

The following is a representative sample of duality results in this vein. To the author's knowledge, first proofs of this particular result date back to Grosberg and Krein [11], and Ellis [10]:

**Theorem 1.2.** *Let  $\alpha \geq 1$ . Let  $X$  be real Banach space, pre-ordered by a closed cone  $X_+$ . Let the dual space  $X'$  be pre-ordered by the dual cone  $X'_+ := \{\phi \in X' \mid \phi(X_+) \subseteq \mathbb{R}_{\geq 0}\}$ , where  $\mathbb{R}_{\geq 0} := \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$ .*

- (1) *The following are equivalent:*
  - (a) *If  $x, a, b \in X$  satisfy  $a \leq x \leq b$ , then  $\|x\| \leq \alpha \max\{\|a\|, \|b\|\}$ .*
  - (b) *For every  $\phi \in X'$ , there exist  $\rho, \psi \in X'_+$  such that  $\phi = \rho - \psi$  and  $\|\rho\| + \|\psi\| \leq \alpha \|\phi\|$ .*
- (2) *The following are equivalent:*
  - (a) *For every  $x \in X$  and  $\beta > \alpha$ , there exist  $a, b \in X_+$  such that  $x = a - b$  and  $\|a\| + \|b\| \leq \beta \|x\|$ .*
  - (b) *If  $\rho, \phi, \psi \in X'$  satisfy  $\rho \leq \phi \leq \psi$ , then  $\|\phi\| \leq \alpha \max\{\|\rho\|, \|\psi\|\}$ .*

Interest in normality and conormality properties is usually traced to their influence on the order structure of the bounded operators between real pre-ordered Banach spaces (cf. [16, Section 3.4] and [5, Section 1.7]).

Further results in the geometric duality theory of real pre-ordered Banach spaces are the duality results between so-called 'additivity properties' (e.g., property (1)(a) in Theorem 1.3 below) and 'coadditivity properties'<sup>3</sup> (e.g., property (2)(a) in Theorem 1.3 below). To the author's knowledge, Theorem 1.3 was first established by Asimow and Ellis [3], [4, Theorems 3.5 and 3.8]. Wong and Ng also established a related result in [21, Lemmas 9.24 and 9.25].

**Theorem 1.3.** *Let  $\alpha \geq 1$  and  $n \in \mathbb{N}$ . Let  $X$  be a real Banach space, pre-ordered by a closed cone  $X_+$ . Let the dual space  $X'$  be pre-ordered by the dual cone  $X'_+ := \{\phi \in X' \mid \phi(X_+) \subseteq \mathbb{R}_{\geq 0}\}$ , where  $\mathbb{R}_{\geq 0} := \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$ .*

<sup>1</sup>The term 'conormality' is due to Walsh [19]. Historically this notion appears under various names in the literature, of which ' $\alpha$ -generating' or 'boundedly generating' is quite common.

<sup>2</sup>The term 'normality' is due to Krein [15].

<sup>3</sup>Historically, this property is called 'directedness'. We prefer the term 'coadditivity' as mnemonic for illustrating the property being dual to notion of 'additivity', which parallels that of the terms 'normality' and 'conormality'.

- (1) *The following are equivalent:*
  - (a) *If  $\{x_j\}_{j=1}^n \subseteq X_+$ , then  $\sum_{j=1}^n \|x_j\| \leq \alpha \|\sum_{j=1}^n x_j\|$ .*
  - (b) *For any  $\{\phi_j\}_{j=1}^n \subseteq X'$ , there exists some  $\varphi \in X'$  such that  $\phi_j \leq \varphi$  for all  $j \in \{1, \dots, n\}$  and  $\|\varphi\| \leq \alpha \max_j \{\|\phi_j\|\}$ .*
- (2) *The following are equivalent:*
  - (a) *For any  $\{x_j\}_{j=1}^n \subseteq X$  and  $\beta > \alpha$ , there exists some  $y \in X$  such that  $x_j \leq y$  for all  $j \in \{1, \dots, n\}$  and  $\|y\| \leq \beta \max_j \{\|x_j\|\}$ .*
  - (b) *If  $\{\phi_j\}_{j=1}^n \subseteq X'_+$ , then  $\sum_{j=1}^n \|\phi_j\| \leq \alpha \|\sum_{j=1}^n \phi_j\|$ .*

In [20] Wickstead established a connection between additivity and coadditivity properties and order-boundedness of norm-precompact sets in real pre-ordered Banach spaces.

Recently, in [7] it was shown that Andô's Theorem (Theorem 1.1, above) is a specific case of a more general result [7, Theorem 4.1]:

**Theorem 1.4.** *Let  $X$  be a real or complex Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$ . If the collection of cones  $\{C_\omega\}_{\omega \in \Omega}$  generate  $X$ , i.e., if every  $x \in X$  can be written as  $x = \sum_{\omega \in \Omega} c_\omega$ , where  $c_\omega \in C_\omega$  for all  $\omega \in \Omega$ , and  $\sum_{\omega \in \Omega} \|c_\omega\| < \infty$ , then there exists a constant  $\alpha \geq 1$  such that every  $x \in X$  can be written as  $x = \sum_{\omega \in \Omega} c_\omega$ , where  $c_\omega \in C_\omega$  for all  $\omega \in \Omega$ , and  $\sum_{\omega \in \Omega} \|c_\omega\| \leq \alpha \|x\|$ .*

*Remark 1.5.* As an aside, we note that [7, Theorem 4.1], in fact, shows that this decomposition can not only be done in a bounded manner, but also (through an application of Michael's Selection Theorem) in a continuous manner: For all  $\omega \in \Omega$ , the maps  $x \mapsto c_\omega$  may be chosen so as to be positively homogeneous, bounded and continuous.

In comparing Andô's Theorem and Theorem 1.4, that the two cones  $X_+$  and  $-X_+$  in a real pre-ordered Banach space  $X$  are related by a minus sign is pure coincidence. What is at play here is how a collection of cones in a Banach space interact, where, in a real pre-ordered Banach space, we merely happen to study interactions of collections of cones comprised of copies of  $X_+$  and  $-X_+$ . As Theorem 1.4 shows, a version of Andô's Theorem still holds regardless of whether the cones are related by minus signs, granted that the entire collection of cones generates the space.

Given the analogous form of the conclusion of Theorem 1.4 to the classical conormality property (2)(a) from Theorem 1.2, a natural question to ask is whether classical geometric duality theory for real pre-ordered Banach spaces (e.g., Theorems 1.2 and 1.3) carries over to the more general situation of Theorem 1.4. Establishing this is the goal of the rest of this paper.

We follow a purely geometric route to this end. We give four general geometric conditions (listed below) on interactions between two cones  $C$  and  $D$  and two convex sets  $B_1$  and  $B_2$  containing zero in a real vector space  $Y$ . The space  $Y$  will be said to be *normal*, *additive*, *conormal* or *coadditive* with respect to  $(C, D, B_1, B_2)$ , if respectively (1), (2), (3) or (4) in the following list of inclusions hold:

- (1)  $(B_2 + C) \cap D \subseteq B_1$
- (2)  $(B_2 \cap C) + D \subseteq B_1$
- (3)  $B_1 \subseteq (B_2 \cap C) + D$
- (4)  $B_1 \subseteq (B_2 + C) \cap D$

Depending on the choice of  $Y$  and the sets  $C, D, B_1$  and  $B_2$ , the properties listed in Theorems 1.2 and 1.3 can be shown to be equivalent to one of the above

set inclusions holding true, e.g., a real pre-ordered Banach space  $X$  satisfies the normality property (1)(a) from Theorem 1.2 if and only if the  $\ell^\infty$ -direct sum  $Y := X \oplus_\infty X$  is normal with respect to  $(C, D, B_1, B_2)$ , with  $C := X_+ \oplus_\infty (-X_+)$ ,  $D := \{(x, x) \in X \oplus_\infty X \mid x \in X\}$ ,  $B_1 := \{(x, x) \in X \oplus_\infty X \mid x \in X, \|x\| \leq \alpha\}$  and  $B_2 := \{(x, y) \in X \oplus_\infty X \mid \|(x, y)\|_\infty \leq 1\}$ . Similarly, the remaining properties from Theorems 1.2 and 1.3 can be shown to be equivalent to one of the above set inclusions for specific choices of  $Y$  and the sets  $C, D, B_1$  and  $B_2$ . Furthermore, this definition is also general enough to encompass more general properties than those that occur in real pre-ordered Banach spaces, e.g., the conclusion of Theorem 1.4 is equivalent to the  $\ell^1$ -direct sum of  $|\Omega|$  copies of  $X$ ,  $Y := \ell^1(\Omega, X)$ , being conormal with respect to  $(C, D, B_1, B_2)$ , where  $C := \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^1(\Omega, X)$ ,  $D := \{\xi \in \ell^1(\Omega, X) \mid \sum_{\omega \in \Omega} \xi_\omega = 0\}$ ,  $B_1 := \{\xi \in \ell^1(\Omega, X) \mid \|\sum_{\omega \in \Omega} \xi_\omega\| \leq 1\}$  and  $B_2 := \{\xi \in \ell^1(\Omega, X) \mid \|\xi\|_1 \leq \alpha\}$ .

We show, using Lemma 3.3 (an elementary application of the one-sided polar calculus) and the conditions outlined in the General Duality Theorems (Theorems 3.4 and 3.5), that normality (additivity) and conormality (coadditivity), as defined above, are dual notions. I.e., for a dual pair of real vector spaces  $(Y, Z)$  with  $C, D, B_1$  and  $B_2$  subsets of  $Y$  as above, we give conditions under which  $Y$  is normal (additive) with respect to  $(C, D, B_1, B_2)$  if and only if  $Z$  is conormal (coadditive) with respect to the one-sided polars  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ , and vice versa.

The General Duality Theorems (Theorems 3.4 and 3.5) hence provide a general framework for proving results in the geometric duality theory for cones in real Banach spaces (and real pre-ordered Banach spaces as a specific case). To apply these results, one is only required to find a relevant dual pair of real vector spaces  $(Y, Z)$  and sets  $C, D, B_1$  and  $B_2$  in  $Y$ , calculate their one-sided polars in  $Z$ , and verify the hypotheses of one of the General Duality Theorems (Theorems 3.4 and 3.5) to obtain a duality result in the vein of Theorems 1.2 and 1.3.

Following this program, generalizing Theorems 1.2 and 1.3 above to real Banach spaces endowed with arbitrary collections of closed cones then becomes a matter of routine (cf. Section 4). We explicitly state our Corollaries 4.7 and 4.9 of Theorems 4.6 and 4.8, which directly generalize Theorems 1.2 and 1.3 above:

**Corollary 4.7.** *Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$  and  $\alpha \geq 1$ .*

- (1) *The following are equivalent:*
  - (a) *If  $\xi \in \mathbf{c}(\Omega, X)$  and  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ , then  $\|x\| \leq \alpha \|\xi\|_\infty$ .*
  - (b) *For every  $\phi \in X'$ , there exists an element  $\eta \in \bigoplus_{\omega \in \Omega} C_\omega^\circ \subseteq \ell^1(\Omega, X')$  such that  $\phi = \sum_{\omega \in \Omega} \eta_\omega$  and  $\|\eta\|_1 \leq \alpha \|\phi\|$ .*
- (2) *The following are equivalent:*
  - (a) *For any  $x \in X$  and  $\beta > \alpha$ , there exists an element  $\xi \in \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^1(\Omega, X)$  such that  $x = \sum_{\omega \in \Omega} \xi_\omega$  and  $\|\xi\|_1 \leq \beta \|x\|$ .*
  - (b) *If  $\eta \in \ell^\infty(\Omega, X')$  and  $\phi \in \bigcap_{\omega \in \Omega} (\eta + C_\omega^\circ)$ , then  $\|\phi\| \leq \alpha \|\eta\|_\infty$ .*

*If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:*

- (3) *The following are equivalent:*
  - (a) *If  $\xi \in \ell^p(\Omega, X)$  and  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ , then  $\|x\| \leq \alpha \|\xi\|_p$ .*
  - (b) *For every  $\phi \in X'$ , there exists an element  $\eta \in \bigoplus_{\omega \in \Omega} C_\omega^\circ \subseteq \ell^q(\Omega, X')$  such that  $\phi = \sum_{\omega \in \Omega} \eta_\omega$  and  $\|\eta\|_q \leq \alpha \|\phi\|$ .*

- (4) The following are equivalent:
- (a) For every  $x \in X$  and  $\beta > \alpha$ , there exists an element  $\xi \in \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^p(\Omega, X)$  such that  $x = \sum_{\omega \in \Omega} \xi_\omega$  and  $\|\xi\|_p \leq \beta \|x\|$ .
  - (b) If  $\eta \in \ell^q(\Omega, X')$  and  $\phi \in \bigcap_{\omega \in \Omega} (\eta + C_\omega^\odot)$ , then  $\|\phi\| \leq \alpha \|\eta\|_q$ .

**Corollary 4.9.** Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$  and  $\alpha \geq 1$ .

- (1) The following are equivalent:
- (a) If  $\xi \in \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^1(\Omega, X)$ , then  $\|\xi\|_1 \leq \alpha \|\sum_{\omega \in \Omega} \xi_\omega\|$ .
  - (b) For every  $\eta \in \ell^\infty(\Omega, X')$ , there exists some  $\phi \in \bigcap_{\omega \in \Omega} (\eta_\omega - C_\omega^\odot)$  with  $\|\phi\| \leq \alpha \|\eta\|_\infty$ .
- (2) The following are equivalent:
- (a) For every  $\xi \in \mathbf{c}(\Omega, X)$  and  $\beta > \alpha$ , there exists some  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega - C_\omega)$  with  $\|x\| \leq \beta \|\xi\|_\infty$ .
  - (b) If  $\eta \in \bigoplus_{\omega \in \Omega} C_\omega^\odot \subseteq \ell^1(\Omega, X')$ , then  $\|\eta\|_1 \leq \alpha \|\sum_{\omega \in \Omega} \eta_\omega\|$ .

If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:

- (3) The following are equivalent:
- (a) If  $\xi \in \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^p(\Omega, X)$ , then  $\|\xi\|_p \leq \alpha \|\sum_{\omega \in \Omega} \xi_\omega\|$ .
  - (b) For every  $\eta \in \ell^q(\Omega, X')$ , there exists some  $\phi \in \bigcap_{\omega \in \Omega} (\eta_\omega - C_\omega^\odot)$  with  $\|\phi\| \leq \alpha \|\eta\|_q$ .
- (4) The following are equivalent:
- (a) For every  $\xi \in \ell^p(\Omega, X)$  and  $\beta > \alpha$ , there exists some  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega - C_\omega)$  with  $\|x\| \leq \beta \|\xi\|_p$ .
  - (b) If  $\eta \in \bigoplus_{\omega \in \Omega} C_\omega^\odot \subseteq \ell^q(\Omega, X')$ , then  $\|\eta\|_q \leq \alpha \|\sum_{\omega \in \Omega} \eta_\omega\|$ .

We note that there are situations where spaces endowed with multiple cones sometimes arise quite naturally. A very simple example is that of a  $C^*$ -algebra  $A$ , when viewed as a real Banach space, which is generated by the four cones  $\{A_+, -A_+, iA_+, -iA_+\}$ , where  $A_+$  is the usual cone of positive elements of  $A$ . In Section 5 we include a brief application of our results from the preceding sections to analyze the geometric properties of naturally occurring cones in  $C^*$ -algebras, culminating in Theorem 5.3 below.

**Theorem 5.3.** Let  $A$  be a  $C^*$ -algebra and let  $A'$  be its dual. Let  $A_+$  denote the cone of positive elements in  $A$ , and let  $A'_+$  denote the cone of positive functionals on  $A$ , defined in the usual way for  $C^*$ -algebras. Let  $A_{\text{sa}}$  and  $A'_{\text{sa}}$  respectively denote the real subspaces of self-adjoint elements in  $A$  and  $A'$ . Then:

- (1) For  $b_1, b_2, b_3, b_4 \in A$ , if
- $$a \in (b_1 + A_+ + iA_{\text{sa}}) \cap (b_2 - A_+ + iA_{\text{sa}}) \\ \cap (b_3 + iA_+ + A_{\text{sa}}) \cap (b_4 - iA_+ + A_{\text{sa}})$$
- then  $\|a\| \leq \max\{\|b_1\|, \|b_2\|\} + \max\{\|b_3\|, \|b_4\|\}$ .
- (2) For  $b_1, b_2, b_3, b_4 \in A$ , if
- $$a \in (b_1 + A_+) \cap (b_2 - A_+) \cap (b_3 + iA_+) \cap (b_4 - iA_+),$$
- then  $\|a\| \leq \max\{\|b_1\|, \|b_2\|\} + \max\{\|b_3\|, \|b_4\|\}$ .
- (3) For  $a, b, c \in A$ , if  $a \leq b \leq c$ , then  $\|b\| \leq 2 \max\{\|a\|, \|c\|\}$ .
- (4) For  $a, b, c \in A_{\text{sa}}$ , if  $a \leq b \leq c$ , then  $\|b\| \leq \max\{\|a\|, \|c\|\}$ .

(5) For  $\phi_1, \phi_2, \phi_3, \phi_4 \in A'$ , if

$$\begin{aligned} \varphi \in & (\phi_1 + A'_+ + iA'_{\text{sa}}) \cap (\phi_2 - A'_+ + iA'_{\text{sa}}) \\ & \cap (\phi_3 + iA'_+ + A'_{\text{sa}}) \cap (\phi_4 - iA'_+ + A'_{\text{sa}}) \end{aligned}$$

then  $\|\varphi\| \leq \sum_{j=1}^4 \|\phi_j\|$ .

(6) For  $\phi_1, \phi_2, \phi_3, \phi_4 \in A'$ , if

$$\varphi \in (\phi_1 + A'_+) \cap (\phi_2 - A'_+) \cap (\phi_3 + iA'_+) \cap (\phi_4 - iA'_+),$$

then  $\|\varphi\| \leq \sum_{j=1}^4 \|\phi_j\|$ .

(7) For  $\rho, \phi, \psi \in A'$ , if  $\rho \leq \phi \leq \psi$ , then  $\|\phi\| \leq 2(\|\rho\| + \|\psi\|)$ .

(8) For  $\rho, \phi, \psi \in A'_{\text{sa}}$ , if  $\rho \leq \phi \leq \psi$ , then  $\|\phi\| \leq \|\rho\| + \|\psi\|$ .

We will now describe the structure of this paper:

In Section 2 we give some preliminary notation and results. We begin, in Section 2.1, with some elementary results on the polar calculus for dual pairs of real vector spaces. Specifically, a result we will use numerous times is Lemma 2.1(8) and (9), giving the exact forms of the one-sided polar of the intersection (sum) of a (closed) convex set containing zero and a cone. Further, in Section 2.2, we give basic definitions and a few basic results on convex analysis, focusing on convex series. These results will be needed in the proofs of the General Duality Theorems (Theorems 3.4 and 3.5).

In Section 3 we give general definitions of normality, additivity, conormality and coadditivity as the four set-inclusions above, and prove our two main results: The General Duality Theorems (Theorems 3.4 and 3.5).

Section 4 serves as an application of the section preceding it. By using the General Duality Theorems (Theorems 3.4 and 3.5), Corollaries 4.7 and 4.9 are established. As already mentioned, results from this section follow as a matter of routine verifications: For a real Banach space  $X$  and arbitrary collection of closed cones  $\{C_\omega\}_{\omega \in \Omega}$ , we choose a related dual pair of real vector spaces  $(Y, Z)$  and sets  $(C, D, B_1, B_2)$  in  $Y$ , compute their polars  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$  in  $Z$ , and verify the hypotheses of the General Duality Theorems (Theorems 3.4 and 3.5) to obtain Theorems 4.6 and 4.8. These theorems are reformulated into Corollaries 4.7 and 4.9 which generalize the classical results Theorems 1.2 and 1.3.

Finally, in Section 5 we give a brief application of our results from the preceding sections to  $C^*$ -algebras. We prove Theorem 5.3 on the geometric structure of naturally occurring cones in  $C^*$ -algebras and their duals.

## 2. PRELIMINARY DEFINITIONS, NOTATION AND RESULTS

All vector spaces in the rest of this paper are assumed to be over  $\mathbb{R}$ . All topological vector spaces are assumed to be Hausdorff. Let  $V$  be a topological vector space with topology  $\tau$ . Let  $A \subseteq V$ . We denote the closure of  $A$  by  $\bar{A}$  (or  $\bar{A}^\tau$  if confusion could arise). The (closed) convex hull of  $A$  will be denoted by  $\text{co}A$  ( $\mathbf{co}A$ ). The topological dual of  $V$  will be denoted by  $V'$ , or by  $(V, \tau)'$  if confusion could arise. A non-empty subset  $C \subseteq V$  will be called a *cone* if  $C + C \subseteq C$  and  $\lambda C \subseteq C$  for all  $\lambda \geq 0$ . If  $C \subseteq V$  is a cone, we define its *dual cone* by  $C' := \{\phi \in V' \mid \phi(C) \subseteq \mathbb{R}_{\geq 0}\}$ , where we denote the non-negative real numbers by  $\mathbb{R}_{\geq 0} := \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$ .

**2.1. The polar calculus.** The current section will give some notation and basic results regarding the one-sided polar calculus. Lemma 2.1(8) and (9) on the one-sided polars of sums (intersections) of (closed) convex sets and cones will be used numerous times in subsequent sections.

Let  $Y$  and  $Z$  be real vector spaces and  $\langle \cdot | \cdot \rangle : Y \times Z \rightarrow \mathbb{R}$  a bilinear map such that  $\{\langle \cdot | z \rangle \mid z \in Z\}$  and  $\{\langle y | \cdot \rangle \mid y \in Y\}$  separate the points of  $Y$  and  $Z$  respectively. We will then call  $(Y, Z)$  a *dual pair*, and the map  $\langle \cdot | \cdot \rangle : Y \times Z \rightarrow \mathbb{R}$  a *duality*. Unless otherwise mentioned,  $Y$  and  $Z$  will be assumed to be respectively endowed with the weak topology (denoted  $\sigma(Y, Z)$ ), and the weak\* topology (denoted  $\sigma(Z, Y)$ ). It is a well-known fact that  $Y' = Z$  and  $Z' = Y$  (cf. [1, Theorem 5.93]).

Let  $(Y, Z)$  be a dual pair,  $A \subseteq Y$  and  $B \subseteq Z$ . We define the *one-sided polar* of  $A$  and  $B$  respectively by  $A^\circ := \{z \in Z \mid \langle a | z \rangle \leq 1, \forall a \in A\}$  and  $B^\circ := \{y \in Y \mid \langle y | b \rangle \leq 1, \forall b \in B\}$ .

We state the following elementary properties of one-sided polars:

**Lemma 2.1.** *Let  $(Y, Z)$  be a dual pair,  $A$  a non-empty subset of  $Y$ ,  $C \subseteq Y$  a cone, and  $\{A_i\}_{i \in I}$  a collection of non-empty subsets of  $Y$ . Then,*

- (1) *The set  $A^\circ$  is closed, convex and contains zero.*
- (2)  *$A \subseteq B$  implies  $A^\circ \supseteq B^\circ$ .*
- (3) *For every  $\lambda > 0$ ,  $(\lambda A)^\circ = \lambda^{-1}(A^\circ)$ .*
- (4)  *$(\bigcup_{i \in I} A_i)^\circ = \bigcap_{i \in I} A_i^\circ$ .*
- (5)  *$A^{\circ\circ} = \mathbf{co}(A \cup \{0\})$ .*
- (6) *If, for every  $i \in I$ ,  $A_i$  is closed convex and contains zero, then*

$$\left(\bigcap_{i \in I} A_i\right)^\circ = \mathbf{co}\left(\bigcup_{i \in I} A_i^\circ\right).$$

- (7)  *$C^\circ \subseteq Z$  is a closed cone and  $C^\circ = -C'$ .*
- (8) *If  $A$  is closed convex and contains zero and  $C$  is closed, then  $(A \cap C)^\circ = \overline{(A^\circ + C^\circ)}$ .*
- (9) *If  $A$  is convex and contains zero, then  $(A + C)^\circ = A^\circ \cap C^\circ$ .*

*Proof.* Proofs of the statements (1)–(4) are elementary and left as an exercise for the reader; (5) is The Bipolar Theorem [1, Theorem 5.103].

We prove (6). It is clear that  $\mathbf{co}(\bigcup_{i \in I} A_i^\circ) \subseteq (\bigcap_{i \in I} A_i)^\circ$ . We prove the reverse inclusion. Let  $z \in (\bigcap_{i \in I} A_i)^\circ$ , but suppose that  $z \notin \mathbf{co}(\bigcup_{i \in I} A_i^\circ)$ . Then by The Separation Theorem [6, Corollary IV.3.10], there exists some  $y \in Y$  and  $\alpha \in \mathbb{R}$  with  $\langle y | \mathbf{co}(\bigcup_{i \in I} A_i^\circ) \rangle < \alpha < \langle y | z \rangle$ . Since, by (1),  $0 \in \mathbf{co}(\bigcup_{i \in I} A_i^\circ)$ , we have  $\alpha > 0$ , and hence, by dividing, we may assume  $\alpha = 1$ , so that  $\langle y | \mathbf{co}(\bigcup_{i \in I} A_i^\circ) \rangle < 1 < \langle y | z \rangle$ . We conclude that  $y \in A_i^{\circ\circ}$  for all  $i \in I$ . Moreover, each  $A_i$  is closed, convex and contains zero, so that, by (6),  $y \in A_i^{\circ\circ} = A_i$  for all  $i \in I$ , and hence  $y \in \bigcap_{i \in I} A_i$ . But  $\langle y | z \rangle > 1$  contradicts the assumption that  $z \in (\bigcap_{i \in I} A_i)^\circ$ .

We prove (7). That  $C^\circ$  is closed follows from (1). Let  $z \in C^\circ$ , i.e.,  $\langle y | z \rangle \leq 1$  for all  $y \in C$ . Since  $\lambda C \subseteq C$  for all  $\lambda \geq 0$ , we conclude that we must have  $\langle y | z \rangle \leq 0$  for all  $y \in C$ . It is now clear that  $C^\circ + C^\circ \subseteq C^\circ$ ,  $\lambda C^\circ \subseteq C^\circ$  for all  $\lambda \geq 0$ , and that  $C^\circ = -C'$ .

We prove (8). By (6),  $(A \cap C)^\circ = \mathbf{co}(A^\circ \cup C^\circ)$ . Since both  $A^\circ$  and  $C^\circ$  contain zero, we have  $A^\circ \cup C^\circ \subseteq A^\circ + C^\circ$ . Furthermore, since  $\overline{(A^\circ + C^\circ)}$  is closed and convex, it is clear that  $\mathbf{co}(A^\circ \cup C^\circ) \subseteq \overline{(A^\circ + C^\circ)}$ . We prove the reverse inclusion

by showing  $(A^\circ + C^\circ) \subseteq (A \cap C)^\circ = \mathbf{co}(A^\circ \cup C^\circ)$ . Indeed, if  $a \in A^\circ$  and  $c \in C^\circ$ , then for every  $y \in A \cap C$ , by (7), we have  $\langle y | a + c \rangle = \langle y | a \rangle + \langle y | c \rangle \leq 1 + 0 = 1$ , so that  $a + c \in (A \cap C)^\circ = \mathbf{co}(A^\circ \cup C^\circ)$ . Therefore  $(A^\circ + C^\circ) \subseteq \mathbf{co}(A^\circ \cup C^\circ) \subseteq \overline{(A^\circ + C^\circ)}$ , and hence  $\mathbf{co}(A^\circ \cup C^\circ) = \overline{(A^\circ + C^\circ)}$ .

We prove (9). Keeping (7) in mind,  $(A + C)^\circ \supseteq A^\circ \cap C^\circ$  follows. Since both  $A$  and  $C$  contain zero, we obtain  $A \cup C \subseteq A + C$ . Then, by (2) and (4),  $(A + C)^\circ \subseteq (A \cup C)^\circ = A^\circ \cap C^\circ$ .  $\square$

**2.2. Convex series.** The current subsection gives basic definitions and results concerning a more general notion of convexity in topological vector spaces; particularly sets that are well behaved with respect to the taking of convex series of their elements (in contrast to *finite* convex combinations where topology does not come into play).

The somewhat technical result, Lemma 2.3(6), will be an essential ingredient in parts of our General Duality Theorems (Theorems 3.4(2)(b) and 3.5(2)(b)). The proofs of many classical results for real pre-ordered Banach spaces, like Theorems 1.2 and 1.3, rely on similar results (cf. [4, Corollary 1.3.3] and [5, Lemma 1.1.3]).

Our terminology follows that of Jameson's from [13, 14]. The terminology of using the prefix "cs" (for convex series) is fairly standard (cf. [14, 22]), although the term " $\sigma$ -convexity" also does occur (cf. [5]).

**Definition 2.2.** Let  $V$  be a topological vector space with topology  $\tau$  and  $A \subseteq V$ .

- (1) The set  $A \subseteq V$  will be called  $\tau$ -pre-cs-compact, if, for all sequences  $\{a_n\} \subseteq A$  and  $\{\lambda_n\} \subseteq \mathbb{R}_{\geq 0}$  with  $\sum_{n=1}^{\infty} \lambda_n = 1$ , the series  $\sum_{n=1}^{\infty} \lambda_n a_n$  converges in the  $\tau$ -topology.
- (2) The set  $A \subseteq V$  will be called  $\tau$ -cs-compact, if, for all sequences  $\{a_n\} \subseteq A$  and  $\{\lambda_n\} \subseteq \mathbb{R}_{\geq 0}$  with  $\sum_{n=1}^{\infty} \lambda_n = 1$ , the series  $\sum_{n=1}^{\infty} \lambda_n a_n$  converges to a point in  $A$  in the  $\tau$ -topology.
- (3) The set  $A \subseteq V$  will be called  $\tau$ -cs-closed if, for sequences  $\{a_n\} \subseteq A$  and  $\{\lambda_n\} \subseteq \mathbb{R}_{\geq 0}$  with  $\sum_{n=1}^{\infty} \lambda_n = 1$ , convergence of the series  $\sum_{n=1}^{\infty} \lambda_n a_n$  in the  $\tau$ -topology implies  $\sum_{n=1}^{\infty} \lambda_n a_n \in A$ .

If no confusion arises as to which topology on  $V$  is meant, we will merely say  $A$  is pre-cs-compact, cs-compact or cs-closed.

The following results give basic properties of pre-cs-compact, cs-compact and cs-closed sets. Most results are elementary and will be left as exercises (references are however given). The result (5) below is a slight generalization of [5, Lemma 1.1.3].

**Lemma 2.3.** *Let  $V$  be a topological vector space.*

- (1) *In  $V$ , every cs-compact set is both cs-closed and pre-cs-compact, and every subset of a pre-cs-compact set is itself pre-cs-compact.*
- (2) *In  $V$ , the intersection of a cs-compact set with a cs-closed set is again cs-compact.*
- (3) *In  $V$ , every closed convex set is cs-closed and every open convex set is cs-closed.*
- (4) *If  $A \subseteq V$  is cs-compact and  $B \subseteq V$  is cs-closed, then  $\mathbf{co}(A \cup B)$  and  $A + B$  are cs-closed.*



- (5) If the topology on  $V$  is normable,  $V$  is a Banach space if and only if its closed unit ball is cs-compact.
- (6) Let  $A \subseteq V$  be cs-closed and let  $G \subseteq D \subseteq \overline{A}$ . If  $G$  is pre-cs-compact such that, for every  $r > 0$  and  $d \in D$ , the set  $(d - rG) \cap A$  is non-empty, then  $D \subseteq \alpha A$  for all  $\alpha > 1$ .

*Proof.* The assertions (1) and (2) follow immediately from the definitions.

Proof of the assertion (3) can be found in [22, Proposition 1.2.1.(i)]. The argument for closed convex sets is elementary. The argument for open convex is slightly more involved and relies on The Separation Theorem [6, Theorem IV.3.7].

An elementary argument will prove (4). Proof can be found in [13, Theorem A.2].

To establish (5), it can be seen that absolutely convergent series converge if and only if the closed unit ball is cs-compact.

We prove (6). Let  $y \in D$  and  $r \in (0, 1)$  be arbitrary. We inductively define sequences  $\{b_n\} \subseteq D$  and  $\{a_n\} \subseteq A$  as follows: For any  $n \in \mathbb{N}$ , if  $\{a_j \mid j = 1, \dots, n-1\} \subseteq A$ , we define  $b_n := r^{-(n-1)}y - \sum_{j=1}^{n-1} r^{j-n}a_j$ . If  $b_n \in D$ , choosing  $a_n \in (b_n - rG) \cap A \neq \emptyset$  then yields

$$\begin{aligned}
 D \supseteq G & \ni r^{-1}b_n - r^{-1}a_n \\
 &= r^{-1} \left( r^{-(n-1)}y - \sum_{j=1}^{n-1} r^{j-n}a_j \right) - r^{n-(n+1)}a_n \\
 &= r^{-n}y - \sum_{j=1}^{n-1} r^{j-(n+1)}a_j - r^{n-(n+1)}a_n \\
 &= r^{-n}y - \sum_{j=1}^n r^{j-(n+1)}a_j \\
 &= b_{n+1}
 \end{aligned}$$

Since  $b_1 = y \in D$ , we therefore obtain the sequences  $\{b_n\} \subseteq D$  and  $\{a_n\} \subseteq A$ .

Since  $b_{n+1} \in G$  for all  $n \in \mathbb{N}$ , and  $G$  is pre-cs-compact, the series  $r^{-1}(r - 1) \sum_{n=1}^{\infty} r^n b_{n+1}$  converges, and hence  $r^n b_{n+1} \rightarrow 0$ . Because  $r^n b_{n+1} = y - \sum_{j=1}^n r^{j-1}a_j$ , we conclude that the series  $\sum_{j=0}^{\infty} r^j a_{j+1}$  converges to  $y$ . Since  $\{a_n\} \subseteq A$  and  $A$  is cs-closed,  $(1 - r) \sum_{j=0}^{\infty} r^j a_{j+1}$  converges a point in  $A$  which must equal  $(1 - r)y$ . We obtain  $y \in (1 - r)^{-1}A$ , and since  $r \in (0, 1)$  was chosen arbitrarily, the result follows.  $\square$

**Lemma 2.4.** *Let  $Y$  be a locally convex space. If  $A \subseteq Y$  is a closed convex set containing zero, then*

$$\bigcap_{\lambda > 1} \lambda A = A.$$

*Proof.* Let  $a \in A$ . If  $a = 0$ , then clearly  $a \in \bigcap_{\lambda > 1} \lambda A$ . Suppose that  $a \neq 0$ . Let  $\lambda > 1$  be arbitrary, so that, by convexity of  $\lambda A$ ,  $a = \lambda^{-1}(\lambda a) + (1 - \lambda^{-1})0 \in \lambda A$ . We conclude that  $A \subseteq \bigcap_{\lambda > 1} \lambda A$ .

We prove the reverse inequality. Suppose  $y \in \bigcap_{\lambda > 1} \lambda A$  is not an element of  $A$ . By The Separation Theorem [6, Theorem IV.3.9], there exists a functional  $\phi \in Y'$  and  $\alpha \in \mathbb{R}$  such that  $\phi(a) < \alpha < \phi(y)$  for all  $a \in A$ . Since  $0 \in A$ , we have that  $\alpha > 0$ . Let  $\lambda_0 := (2\alpha)^{-1}(\alpha + \phi(y)) > 1$ . For every  $a \in A$ ,

$$\phi(\lambda_0 a) = \frac{(\alpha + \phi(y))\phi(a)}{2\alpha} < \frac{(\alpha + \phi(y))\alpha}{2\alpha} = \frac{\alpha + \phi(y)}{2} < \phi(y).$$

Therefore  $\phi(y - \lambda_0 a) > 0$  for every  $a \in A$ . Hence  $y \notin \lambda_0 A$ , contradicting the assumption  $y \in \bigcap_{\lambda > 1} \lambda A$ . Therefore  $\bigcap_{\lambda > 1} \lambda A \subseteq A$ . We conclude that  $\bigcap_{\lambda > 1} \lambda A = A$ .  $\square$

### 3. GEOMETRIC DUALITY THEORY FOR CONES IN DUAL PAIRS OF VECTOR SPACES: A GENERAL FRAMEWORK

In this section we define general notions of normality, additivity, conormality and coadditivity as interactions of two cones with two convex sets containing zero. Using our preliminary results from the previous section we prove our main results: The General Duality Theorems (Theorems 3.4 and 3.5).

**Definition 3.1.** Let  $Y$  be a vector space. Let  $C, D \subseteq Y$  be cones and  $B_1, B_2 \subseteq Y$  convex sets containing zero.

- (1) We will say that  $Y$  is *normal* with respect to  $(C, D, B_1, B_2)$  if

$$(B_2 + C) \cap D \subseteq B_1.$$

- (2) We will say that  $Y$  is *additive* with respect to  $(C, D, B_1, B_2)$  if

$$(B_2 \cap C) + D \subseteq B_1.$$

- (3) We will say that  $Y$  is *conormal* with respect to  $(C, D, B_1, B_2)$  if

$$B_1 \subseteq (B_2 \cap C) + D.$$

- (4) We will say that  $Y$  is *coadditive* with respect to  $(C, D, B_1, B_2)$  if

$$B_1 \subseteq (B_2 + C) \cap D.$$

*Remark 3.2.* We note that, if, for some  $\alpha > 0$ ,  $Y$  has one of the above properties with respect to  $(C, D, B_1, \alpha B_2)$ , then it has the same property with respect to  $(C, D, \alpha^{-1} B_1, B_2)$ .

Elementary applications of Lemma 2.1 yield the following result.

**Lemma 3.3.** Let  $(Y, Z)$  be a dual pair with  $\sigma(Y, Z)$ -closed cones  $C, D \subseteq Y$  and  $B_1, B_2 \subseteq Y$   $\sigma(Y, Z)$ -closed convex sets containing zero.

- (1) If  $Y$  is normal with respect to  $(C, D, B_1, B_2)$ , then  $B_1^\circ \subseteq \overline{(B_2^\circ \cap C^\circ) + D^\circ}$ .
- (2) If  $Y$  is additive with respect to  $(C, D, B_1, B_2)$ , then  $B_1^\circ \subseteq \overline{(B_2^\circ + C^\circ) \cap D^\circ}$ .
- (3) If  $Y$  is conormal with respect to  $(C, D, B_1, B_2)$ , then  $(B_2^\circ + C^\circ) \cap D^\circ \subseteq B_1^\circ$ .
- (4) If  $Y$  is coadditive with respect to  $(C, D, B_1, B_2)$ , then  $(B_2^\circ \cap C^\circ) + D^\circ \subseteq B_1^\circ$ .
- (5) If  $Z$  is normal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ , then  $B_1 \subseteq \overline{B_2 \cap C + D}$ .
- (6) If  $Z$  is additive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ , then  $B_1 \subseteq \overline{(B_2 + C) \cap D}$ .
- (7) If  $Z$  is conormal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ , then  $(B_2 + C) \cap D \subseteq B_1$ .
- (8) If  $Z$  is coadditive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ , then  $(B_2 \cap C) + D \subseteq B_1$ .

*Proof.* The results follow from elementary applications of Lemma 2.1 (and noting that for all subsets  $K$  and  $L$  of a topological space, both  $K \subseteq \overline{K}$  and  $K \cap L \subseteq \overline{K} \cap L$  hold).  $\square$

With the above observation and the results established in the previous section, the General Duality Theorems below now become fairly simple verifications.

**Theorem 3.4.** (*General duality between normality and conormality*) Let  $(Y, Z)$  be a dual pair with  $\sigma(Y, Z)$ -closed cones  $C, D \subseteq Y$  and  $B_1, B_2 \subseteq Y$   $\sigma(Y, Z)$ -closed convex sets containing zero.

- (1) Of the statements (i) and (ii) below:
  - (a) (ii) implies (i).
  - (b) If  $B_2^\circ \cap C^\circ + D^\circ$  is  $\sigma(Z, Y)$ -closed, then (i) and (ii) are equivalent.
    - (i)  $Y$  is normal with respect to  $(C, D, B_1, B_2)$ .
    - (ii)  $Z$  is conormal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .
- (2) Of the statements (i)–(iii) below:
  - (a) (ii) implies (iii).
  - (b) If  $B_2 \cap C + D$  is  $\sigma(Y, Z)$ -cs-closed and  $B_1$  contains a  $\sigma(Y, Z)$ -pre-cs-compact set  $G$ , such that, for every  $r > 0$  and  $b \in B_1$ ,  $(b - rG) \cap (B_2 \cap C + D) \neq \emptyset$ , then (ii) and (iii) are equivalent.
  - (c) If  $B_2 \cap C + D$  is  $\sigma(Y, Z)$ -closed, then (i), (ii) and (iii) are equivalent.
    - (i)  $Y$  is conormal with respect to  $(C, D, B_1, B_2)$ .
    - (ii)  $Y$  is conormal with respect to  $(C, D, B_1, \lambda B_2)$  for all  $\lambda > 1$ .
    - (iii)  $Z$  is normal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

*Proof.* We prove (1)(a). This is immediate from Lemma 3.3.

We prove (1)(b). By (1)(a), it suffices to prove that (i) implies (ii). By Lemma 3.3 above and the assumption that  $B_2^\circ \cap C^\circ + D^\circ$  is closed,  $B_1^\circ \subseteq \overline{B_2^\circ \cap C^\circ + D^\circ} = B_2^\circ \cap C^\circ + D^\circ$ . We conclude that  $Z$  is conormal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

We prove (2)(a). By Lemma 3.3 and Remark 3.2,  $(B_2^\circ + C^\circ) \cap D^\circ \subseteq \lambda(B_1^\circ)$  holds for every  $\lambda > 1$ . By Lemma 2.4,  $\bigcap_{\lambda > 1} \lambda B_1^\circ = B_1^\circ$ . Therefore  $(B_2^\circ + C^\circ) \cap D^\circ \subseteq \bigcap_{\lambda > 1} \lambda B_1^\circ = B_1^\circ$ , and we conclude that  $Z$  is normal with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

We prove (2)(b). By (2)(a) it suffices to prove that (iii) implies (ii). By Lemma 3.3 above  $B_1 \subseteq \overline{B_2 \cap C + D}$ . Since  $B_2 \cap C + D$  is cs-closed and  $B_1$  contains a pre-cs-compact set with the stated property, by Lemma 2.3, for every  $\lambda > 1$ ,  $B_1 \subseteq \lambda(B_2 \cap C + D) = (\lambda B_2) \cap C + D$ . We conclude that  $Y$  is conormal with respect to  $(C, D, B_1, \lambda B_2)$  for all  $\lambda > 1$ .

We prove (2)(c). By (2)(a) we have that (ii) implies (iii). Since  $B_2 \subseteq \lambda B_2$  for all  $\lambda > 1$ , we have  $B_1 \subseteq C \cap B_2 + D \subseteq C \cap (\lambda B_2) + D$ , and hence (i) implies (ii). If (iii) holds, then by Lemma 3.3 above, and the assumption that  $B_2 \cap C + D$  is closed,  $B_1 \subseteq \overline{B_2 \cap C + D} = B_2 \cap C + D$ , so that (iii) implies (i).  $\square$

**Theorem 3.5.** (*General duality between additivity and coadditivity*) Let  $(Y, Z)$  be a dual pair with  $\sigma(Y, Z)$ -closed cones  $C, D \subseteq Y$  and  $B_1, B_2 \subseteq Y$   $\sigma(Y, Z)$ -closed convex sets containing zero.

- (1) Of the statements (i) and (ii) below:
  - (a) (ii) implies (i).
  - (b) If  $B_2^\circ + C^\circ$  is  $\sigma(Z, Y)$ -closed, then (i) and (ii) are equivalent.
    - (i)  $Y$  is additive with respect to  $(C, D, B_1, B_2)$ .
    - (ii)  $Z$  is coadditive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

- (2) Of the statements (i)–(iii) below:
- (a) (ii) implies (iii).
  - (b) If  $B_2 + C$  is  $\sigma(Y, Z)$ -cs-closed and  $B_1$  contains a  $\sigma(Y, Z)$ -pre-cs-compact set  $G$  such that, for every  $b \in B_1$  and  $r > 0$ ,  $(b - rG) \cap (B_2 + C) \neq \emptyset$ , then (ii) and (iii) are equivalent.
  - (c) If  $B_2 + C$  is  $\sigma(Y, Z)$ -closed, then (i), (ii) and (iii) are equivalent.
    - (i)  $Y$  is coadditive with respect to  $(C, D, B_1, B_2)$ .
    - (ii)  $Y$  is coadditive with respect to  $(C, D, B_1, \lambda B_2)$  for all  $\lambda > 1$ .
    - (iii)  $Z$  is additive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

*Proof.* We prove (1)(a). This is immediate from Lemma 3.3.

We prove (1)(b). By (1)(a) it suffices to show that (i) implies (ii). By Lemma 3.3,  $B_1^\circ \subseteq \overline{(B_2^\circ + C^\circ)} \cap D^\circ$ . Since  $B_2^\circ + C^\circ$  is closed,  $B_1^\circ \subseteq (B_2^\circ + C^\circ) \cap D^\circ$ , and we conclude that  $Z$  is coadditive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

We prove (2)(a). By Lemma 3.3 and Remark 3.2,  $(B_2^\circ + C^\circ) \cap D^\circ \subseteq \lambda B_1^\circ$  for every  $\lambda > 1$ . By Lemma 2.4,  $(B_2^\circ \cap C^\circ + D^\circ) \subseteq \bigcap_{\lambda > 1} \lambda(B_1^\circ) = B_1^\circ$ . We conclude that  $Z$  is additive with respect to  $(C^\circ, D^\circ, B_1^\circ, B_2^\circ)$ .

We prove (2)(b). By (2)(a) it suffices to prove that (iii) implies (ii). By Lemma 3.3,  $B_1 \subseteq \overline{(B_2 + C)} \cap D \subseteq \overline{(B_2 + C)}$ . Since  $B_1$  is assumed to contain a pre-cs-compact set with the stated property, by Lemma 2.3,  $B_1 \subseteq \lambda(B_2 + C) = \lambda B_2 + C$  holds for every  $\lambda > 1$ . Now, because  $B_1 \subseteq \overline{(B_2 + C)} \cap D \subseteq D$ , we see that  $B_1 \subseteq (\lambda B_2 + C) \cap D$  holds for every  $\lambda > 1$ . We conclude that  $Y$  is coadditive with respect to  $(C, D, B_1, \lambda B_2)$  for all  $\lambda > 1$ .

We prove (2)(c). By 2(a), (ii) implies (iii). Since  $B_2 \subseteq \lambda B_2$  for all  $\lambda > 1$ , we have  $B_1 \subseteq (B_2 + C) \cap D \subseteq (\lambda B_2 + C) \cap D$  for all  $\lambda > 1$ , so that (i) implies (ii). If (iii) holds, by Lemma 3.3 and the hypothesis that  $B_2 + C$  is closed,  $B_1 \subseteq \overline{(B_2 + C)} \cap D = (B_2 + C) \cap D$ , so that (iii) implies (i).  $\square$

*Remark 3.6.* The most technical parts of the two theorems above is the reliance on Lemma 2.3(6) in the proofs of Theorems 3.4(2)(b) and 3.5(2)(b) below, in which a duality result follows by “paying an arbitrarily small price” in scaling up the set  $B_2$ . The necessity of Lemma 2.3(6) can be explained by the set  $B_2$  in the previous two theorems not being  $\sigma(Y, Z)$ -compact in general. If  $B_2$  is  $\sigma(Y, Z)$ -compact, then Theorems 3.4(2)(c) and 3.5(2)(c) apply and Lemma 2.3(6) is not needed. The situation is likely best seen in the setting of Banach spaces, e.g., Theorem 4.6, where  $B_2$  is chosen to be a closed ball of a not-necessarily reflexive Banach space and is hence is not necessarily weakly-compact. In [4, Examples 2.1.6 and 2.3.10] non-reflexive examples are given, showing that the conclusions of Theorems 3.4(2)(b) and 3.5(2)(b) are indeed the best possible.

#### 4. APPLICATION: GEOMETRIC DUALITY THEORY OF CONES IN BANACH SPACES

The current section is a routine application of the General Duality Theorems (Theorems 3.4 and 3.5) from the previous section. Our main results in this section are Theorems 4.6 and 4.8 and their reformulations Corollaries 4.7 and 4.9. These results generalize Theorems 1.2 and 1.3 from the introduction.

We begin with some preliminary notation and definitions used in this section:

For sets  $A$  and  $B$ , by  $B^A$  we will denote the set of all functions from  $A$  to  $B$ . Throughout this section  $X$  will denote an arbitrary real Banach space and  $X'$  its

topological dual. The map  $\langle \cdot | \cdot \rangle : X \times X' \rightarrow \mathbb{R}$  will denote the usual (evaluation) duality for the dual pair  $(X, X')$ . We will denote the closed unit ball of a Banach space  $X$  by  $\mathbf{B}_X$ .

For an index set  $\Omega$  we will denote the directed set (ordered by inclusion) of finite subsets of  $\Omega$  by  $\mathcal{F}(\Omega)$ . For the sake of readability, the value of a function  $\xi \in X^\Omega$  at some  $\omega \in \Omega$  will be usually be denoted by  $\xi_\omega$  instead of  $\xi(\omega)$ . For any function  $f \in \mathbb{R}^\Omega$  and  $x \in X$ , we define  $f \otimes x \in X^\Omega$  by  $\Omega \ni \omega \mapsto f(\omega)x$ . For  $A \subseteq \Omega$  we will denote the characteristic function of  $A$  by  $\chi_A$ , and define  $\delta_\omega := \chi_{\{\omega\}}$  for all  $\omega \in \Omega$ .

For  $\xi \in X^\Omega$ , by  $\sum_{\omega \in \Omega} \xi_\omega$  we mean the norm-limit of the net  $\{\sum_{\omega \in F} \xi_\omega\}_{F \in \mathcal{F}(\Omega)}$  (if it exists). For subspaces  $Y \subseteq X^\Omega$  and  $Z \subseteq X'^\Omega$ , if  $Y \times Z \ni (\xi, \eta) \mapsto \sum_{\omega \in \Omega} \langle \xi_\omega | \eta_\omega \rangle$  defines a duality on  $(Y, Z)$ , we will denote it by  $\langle \cdot | \cdot \rangle : Y \times Z \rightarrow \mathbb{R}$ .

We define the following classical direct sums of a Banach space:

**Definition 4.1.** Let  $\Omega$  be an index set and  $X$  a Banach space.

- (1) For  $1 \leq p < \infty$ , by  $\ell^p(\Omega, X)$  we will denote the subspace of  $X^\Omega$  of all elements  $\xi \in X^\Omega$  satisfying  $\sum_{\omega \in \Omega} \|\xi_\omega\|^p < \infty$ , with norm  $\|\xi\|_p := (\sum_{\omega \in \Omega} \|\xi_\omega\|^p)^{1/p}$ .
- (2) By  $\ell^\infty(\Omega, X)$  we will denote the subspace of  $X^\Omega$  of all elements  $\xi \in X^\Omega$  satisfying  $\sup_{\omega \in \Omega} \|\xi_\omega\| < \infty$ , with norm  $\|\xi\|_\infty := \sup_{\omega \in \Omega} \|\xi_\omega\|$ .
- (3) By  $\mathbf{c}(\Omega, X)$  we will denote the closed subspace of  $\ell^\infty(\Omega, X)$  of all elements  $\xi \in \ell^\infty(\Omega, X)$  for which there exists some  $x \in X$  such that, for every  $\varepsilon > 0$ , there exists some  $F \in \mathcal{F}(\Omega)$ , with  $\sup_{\omega \in \Omega \setminus F} \|\xi_\omega - x\| < \varepsilon$ .

We will use the folklore-result (cf. [8]) that the duals of  $\mathbf{c}(\Omega, X)$ ,  $\ell^1(\Omega, X)$  and  $\ell^p(\Omega, X)$  (for  $1 < p, q < \infty$  with  $p^{-1} + q^{-1} = 1$ ) may be isometrically isomorphically identified with  $\ell^1(\Omega, X')$ ,  $\ell^\infty(\Omega, X')$  and  $\ell^q(\Omega, X')$  respectively, where evaluation is given by  $\langle \cdot | \cdot \rangle$ .

**Definition 4.2.** Let  $X$  be a Banach space and  $\Omega$  an index set.

- (1) We define the *canonical summation operator*  $\Sigma : X^\Omega \rightarrow X \cup \{\infty\}$  as follows:

$$\Sigma \xi := \begin{cases} \sum_{\omega \in \Omega} \xi_\omega & \text{If } \sum_{\omega \in \Omega} \xi_\omega \text{ converges in norm in } X \\ \infty & \text{otherwise.} \end{cases}$$

The set  $D(\Sigma) := \Sigma^{-1}(X) \subseteq X^\Omega$  will be called its domain.

- (2) We define the *constant part operator*  $\mathbf{const} : X^\Omega \rightarrow X \cup \{\infty\}$  as follows: If  $|\Omega| < \infty$ , then, for  $\xi \in X^\Omega$ , we define  $\mathbf{const} \xi := |\Omega|^{-1} \sum_{\omega \in \Omega} \xi_\omega$ . If  $|\Omega| \not< \infty$ , then we define the map  $\mathbf{const}$  as follows: Let  $\xi \in X^\Omega$ . If there exists some  $x \in X$  such that, for any  $\varepsilon > 0$ , there exists some  $F \in \mathcal{F}(\Omega)$ , so that  $\sup_{\omega \in \Omega \setminus F} \|\xi_\omega - x\| < \varepsilon$ , we define  $\mathbf{const} \xi := x$ . If there exists no such  $x$ , we define  $\mathbf{const} \xi := \infty$ . The set  $D(\mathbf{const}) := \mathbf{const}^{-1}(X) \subseteq X^\Omega$  will be called its domain.

**Definition 4.3.** Let  $X$  be a Banach space. For some index set  $\Omega$ , let  $\{C_\omega\}_{\omega \in \Omega}$  a collection of cones in  $X$ . Let  $Y \subseteq X^\Omega$  be a subspace. We define the following sets

- (1)  $(\oplus C)(Y) := \{\xi \in Y \mid \forall \omega \in \Omega, \xi_\omega \in C_\omega\}$
- (2)  $\Sigma_0(Y) := \{\xi \in Y \cap D(\Sigma) \mid \Sigma\xi = 0\}$
- (3)  $\Sigma_1(Y) := \{\xi \in Y \cap D(\Sigma) \mid \|\Sigma\xi\| \leq 1\}$
- (4)  $\Xi_\infty(Y) := \{\xi \in Y \cap D(\mathbf{const}) \mid \chi_\Omega \otimes \mathbf{const}\xi \in Y, \xi = \chi_\Omega \otimes \mathbf{const}\xi\}$
- (5)  $\Xi_1(Y) := \Xi_\infty(Y) \cap \{\xi \in Y \cap D(\mathbf{const}) \mid \|\mathbf{const}\xi\| \leq 1\}$ .

It will often happen that we refer to a collection of sets of the above forms that all occur in a single space  $Y$ . For the sake of readability, we will suppress repeated mention of  $Y$  by introducing the following abbreviation when referring to such a collection. Explicitly, by the phrase

$$“\oplus C, \Sigma_0, \Sigma_1, \Xi_\infty, \Xi_1 \text{ in } Y”,$$

we will mean

$$“(\oplus C)(Y), \Sigma_0(Y), \Sigma_1(Y), \Xi_\infty(Y), \Xi_1(Y)”.$$

To be clear as to our notation, if  $(Y, Z)$  with  $Y \subseteq X^\Omega$  and  $Z \subseteq X'^\Omega$  is a dual pair with respect to  $\langle \cdot, \cdot \rangle$ , we explicitly differentiate the meaning of “ $\oplus C^\circ$  in  $Z$ ”, i.e.,  $(\oplus C^\circ)(Z) = \{\eta \in Z \mid \forall \omega \in \Omega, \eta_\omega \in C_\omega^\circ\}$  – the direct sum of the collection of one-sided polars  $\{C_\omega^\circ\}_{\omega \in \Omega}$ , from “ $(\oplus C)^\circ$ ”, i.e.,  $((\oplus C)(Y))^\circ = \{\eta \in Z \mid \forall \xi \in (\oplus C)(Y), \langle \xi, \eta \rangle \leq 1\}$  – the one-sided polar of the direct sum  $(\oplus C)(Y)$ .

**Lemma 4.4.** Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$ .

- (1) In the dual pair  $(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))$  the one-sided polars of the sets  $\oplus C$ ,  $\Xi_\infty$ ,  $\Xi_1$ , and  $\mathbf{B}_{\mathbf{c}(\Omega, X)}$  in  $\mathbf{c}(\Omega, X)$  respectively equal the sets  $\oplus C^\circ$ ,  $\Sigma_0$ ,  $\Sigma_1$  and  $\mathbf{B}_{\ell^1(\Omega, X')}$  in  $\ell^1(\Omega, X')$ .
- (2) In the dual pair  $(\ell^1(\Omega, X), \ell^\infty(\Omega, X'))$  the one-sided polars of the sets  $\oplus C$ ,  $\Sigma_0$ ,  $\Sigma_1$  and  $\mathbf{B}_{\ell^1(\Omega, X)}$  in  $\ell^1(\Omega, X)$  respectively equal the sets  $\oplus C^\circ$ ,  $\Xi_\infty$ ,  $\Xi_1$  and  $\mathbf{B}_{\ell^\infty(\Omega, X')}$  in  $\ell^\infty(\Omega, X')$ .
- (3) In the dual pair  $(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ , for  $1 \leq p, q \leq \infty$  with  $p^{-1} + q^{-1} = 1$  and  $|\Omega| < \infty$ , the one-sided polars of the sets  $\oplus C$ ,  $\Xi_\infty$ ,  $\Xi_1$ ,  $\Sigma_1$ ,  $\Sigma_0$  and  $\mathbf{B}_{\ell^p(\Omega, X)}$  in  $\ell^p(\Omega, X)$  respectively equal the sets  $\oplus C^\circ$ ,  $\Sigma_0$ ,  $\Sigma_1$ ,  $\Xi_1$ ,  $\Xi_\infty$  and  $\mathbf{B}_{\ell^q(\Omega, X')}$  in  $\ell^q(\Omega, X')$ .

*Proof.* We prove (1):

It is clear that  $\oplus C^\circ \subseteq (\oplus C)^\circ$ . We prove the reverse inclusion. Let  $\eta \in (\oplus C)^\circ \subseteq \ell^1(\Omega, X')$ , but suppose that  $\eta \notin \oplus C^\circ$ . By The Separation Theorem [6, Corollary IV.3.10], there exists some  $\xi \in \mathbf{c}(\Omega, X)$  and  $\alpha \in \mathbb{R}$  such that  $\langle \xi, \eta \rangle > \alpha > \langle \xi, \rho \rangle$  for all  $\rho \in \oplus C^\circ \subseteq \ell^1(\Omega, X')$ . Since  $0 \in \oplus C^\circ$ , we have  $\alpha > 0$ . For every  $\omega \in \Omega$ ,  $\lambda \geq 0$  and  $\phi \in C_\omega^\circ \subseteq X'$  we have  $\delta_\omega \otimes (\lambda\phi) \in \oplus C^\circ$ , and therefore  $\alpha > \langle \xi, \delta_\omega \otimes (\lambda\phi) \rangle = \lambda \langle \xi_\omega, \phi \rangle$  implies  $\langle \xi_\omega, \phi \rangle \leq 0 \leq 1$  for all  $\omega \in \Omega$ . I.e.,  $\xi_\omega \in (C_\omega)^\circ = C_\omega^\circ$  for all  $\omega \in \Omega$ , so that  $\xi \in \oplus C \subseteq \mathbf{c}(\Omega, X)$ . For every  $\lambda \geq 0$ ,  $\lambda\xi \in \oplus C$ , so that  $\langle \lambda\xi, \eta \rangle \leq 1$  implies  $\langle \xi, \eta \rangle \leq 0$ , yielding the absurdity  $0 < \alpha < \langle \xi, \eta \rangle \leq 0$ . We conclude that  $\eta \in \oplus C^\circ$ , and hence that  $\oplus C^\circ = (\oplus C)^\circ$ .

It is clear that  $\Sigma_0 \subseteq \Xi_\infty^\circ$ . We prove the reverse inclusion. Let  $\eta \in \Xi_\infty^\circ$ , but suppose  $\eta \notin \Sigma_0$ . Then  $\Sigma\eta \neq 0$  implies that there exists some  $x \in X$  such that  $\langle x, \Sigma\eta \rangle > 1$ . Since  $\chi_\Omega \otimes x \in \Xi_\infty \subseteq \mathbf{c}(\Omega, X)$ , we then have  $\langle \chi_\Omega \otimes x, \eta \rangle = \langle x, \Sigma\eta \rangle > 1$ , so that  $\eta \notin \Xi_\infty^\circ$ , contradicting our assumption that  $\eta \in \Xi_\infty^\circ$ . We conclude  $\Xi_\infty^\circ \subseteq \Sigma_0$ .

It is clear that  $\Sigma_1 \subseteq \Xi_1^\circ$ . We prove the reverse inclusion. Let  $\eta \in \Xi_1^\circ$ , but suppose that  $\eta \notin \Sigma_1$ . Then there exists some  $x \in \mathbf{B}_X$  such that  $\langle x | \Sigma \eta \rangle > 1$ . As before  $\langle \chi_\Omega \otimes x | \eta \rangle = \langle x | \Sigma \eta \rangle > 1$ , while  $\chi_\Omega \otimes x \in \Xi_1 \subseteq \mathbf{c}(\Omega, X)$ , contradicting our assumption that  $\eta \in \Xi_1^\circ$ . We conclude  $\Xi_1^\circ \subseteq \Sigma_1$ .

Since  $\ell^1(\Omega, X')$  is the dual of  $\mathbf{c}(\Omega, X)$ , it follows that  $\mathbf{B}_{\ell^1(\Omega, X')} = \mathbf{B}_{\mathbf{c}(\Omega, X)}^\circ$ .

We prove (2):

That  $\oplus C^\circ = (\oplus C)^\circ$  follows as in (1).

It is clear that  $\Xi_\infty \subseteq \Sigma_0^\circ$ . We prove the reverse inclusion. Let  $\eta \in \Sigma_0^\circ$ , but suppose  $\eta \notin \Xi_\infty$ . Then there exist  $\omega_0, \omega_1 \in \Omega$  such that  $\eta_{\omega_0} - \eta_{\omega_1} \neq 0$ . Let  $x \in X$  be such that  $\langle x | \eta_{\omega_0} - \eta_{\omega_1} \rangle > 1$ . Then  $\delta_{\omega_0} \otimes x - \delta_{\omega_1} \otimes x \in \Sigma_0$ , and  $1 \geq \langle \delta_{\omega_0} \otimes x - \delta_{\omega_1} \otimes x | \eta \rangle = \langle x | \eta_{\omega_0} - \eta_{\omega_1} \rangle > 1$ , which is absurd. We conclude that  $\Sigma_0^\circ \subseteq \Xi_\infty$ , and hence  $\Sigma_0^\circ = \Xi_\infty$ .

It is clear that  $\Xi_1 \subseteq \Sigma_1^\circ$ . We prove the reverse inclusion. Let  $\eta \in \Sigma_1^\circ$ , but suppose  $\eta \notin \Xi_1$ . Since  $\Sigma_0 \subseteq \Sigma_1$ , by Lemma 2.1,  $\Sigma_1^\circ \subseteq \Sigma_0^\circ = \Xi_\infty$ . Therefore  $\eta \in \Xi_\infty$ , but since  $\eta \notin \Xi_1$ , we have  $\|\mathbf{const}\eta\| > 1$ . Let  $x \in \mathbf{B}_X$  be such that  $\langle x | \mathbf{const}\eta \rangle > 1$ , then, for any  $\omega \in \Omega$ , we have  $\delta_\omega \otimes x \in \Sigma_1$  and  $1 \geq \langle \delta_\omega \otimes x | \eta \rangle = \langle x | \mathbf{const}\eta \rangle > 1$ , which is absurd. We conclude that  $\Xi_1 \subseteq \Sigma_1^\circ$ , and hence  $\Xi_1 = \Sigma_1^\circ$ .

Since  $\ell^\infty(\Omega, X')$  is the dual  $\ell^1(\Omega, X)$ , it follows that  $\mathbf{B}_{\ell^\infty(\Omega, X')} = \mathbf{B}_{\ell^1(\Omega, X)}^\circ$ .

The result (3) follows as in (1) and (2).  $\square$

*Remark 4.5.* If  $|\Omega| \not\leq \infty$  and  $1 < p < \infty$ , then the canonical summation operator  $\Sigma$  on  $D(\Sigma) \cap \ell^p(\Omega, \mathbb{R})$  is an unbounded non-closable operator (and hence also for  $\ell^p(\Omega, X)$  for any Banach space  $X$ ). To see this, consider the sequence  $\{\xi^{(n)}\}_{n \in \mathbb{N}} \subseteq \ell^p(\mathbb{N}, \mathbb{R})$ , defined by  $\xi_j^{(n)} := 2^{-m-n}$  if  $j \in \{(m-1)2^n + 1, (m-1)2^n + 2, \dots, m2^n - 1, m2^n\}$  for all  $j, m, n \in \mathbb{N}$ . Then  $\Sigma \xi^{(n)} = \sum_{j=1}^\infty \xi_j^{(n)} = 2^n \sum_{m=1}^\infty 2^{-m-n} = 1$  for all  $n \in \mathbb{N}$ , while

$$\begin{aligned} \|\xi^{(n)}\|_p^p &= \sum_{j=1}^\infty (\xi_j^{(n)})^p \\ &= 2^{-np+n} \sum_{m=1}^\infty 2^{-mp} \\ &= 2^{-np+n} \frac{2^{-p}}{1 - 2^{-p}} \end{aligned}$$

implies  $\xi^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ . Using this observation, one can show that the norm-closure of  $\Sigma_0$ , and hence also the norm-closure of  $\Sigma_1$ , is all of  $\ell^p(\mathbb{N}, \mathbb{R})$ . Therefore neither  $\Sigma_0$  nor  $\Sigma_1$  is norm-closed, and hence, not weakly closed.

Our General Duality Theorems therefore do not apply when substituting  $\Sigma_0$  for  $C$  or  $D$ , or  $\Sigma_1$  for  $B_1$  or  $B_2$  as in the hypotheses of Theorems 3.4 and 3.5 for the dual pair  $(\ell^p(\Omega, X), \ell^q(\Omega, X'))$  in the case where  $|\Omega| \not\leq \infty$  and  $1 < p < \infty$  with  $p^{-1} + q^{-1} = 1$ .

Having computed the one-sided polars of the required sets in Lemma 3.3, it is now a routine matter to apply our General Duality Theorems (Theorems 3.4 and 3.5) to establish Theorems 4.6 and 4.8 below. Simple verifications (left to the reader) will establish the reformulations of Theorems 4.6 and 4.8 given in Corollaries 4.7 and 4.9. These corollaries can be seen to generalize Theorems 1.2 and 1.3 from the introduction.

**Theorem 4.6.** *Let  $\alpha \geq 1$ . Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$ .*

- (1) *The space  $\mathbf{c}(\Omega, X)$  is normal with respect to  $(\oplus C, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\mathbf{c}(\Omega, X)})$  in  $\mathbf{c}(\Omega, X)$  if and only if the space  $\ell^1(\Omega, X')$  is conormal with respect to  $(\oplus C^\odot, \Sigma_0, \Sigma_1, \alpha \mathbf{B}_{\ell^1(\Omega, X')})$  in  $\ell^1(\Omega, X')$ .*
- (2) *The space  $\ell^1(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^1(\Omega, X)})$  in  $\ell^1(\Omega, X)$  for every  $\beta > \alpha$  if and only if the space  $\ell^\infty(\Omega, X')$  is normal with respect to  $(\oplus C^\odot, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^\infty(\Omega, X')})$  in  $\ell^\infty(\Omega, X')$ .*

*If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:*

- (3) *The space  $\ell^p(\Omega, X)$  is normal with respect to  $(\oplus C, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^p(\Omega, X)})$  in  $\ell^p(\Omega, X)$  if and only if the space  $\ell^q(\Omega, X')$  is conormal with respect to  $(\oplus C^\odot, \Sigma_0, \Sigma_1, \alpha \mathbf{B}_{\ell^q(\Omega, X')})$  in  $\ell^q(\Omega, X')$ .*
- (4) *The space  $\ell^p(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^p(\Omega, X)})$  in  $\ell^p(\Omega, X)$  for every  $\beta > \alpha$  if and only if the space  $\ell^q(\Omega, X')$  is normal with respect to  $(\oplus C^\odot, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^q(\Omega, X')})$  in  $\ell^q(\Omega, X')$ .*

*Proof.* We prove (1). By Lemma 4.4, the one-sided polars of the sets  $\Xi_\infty$ ,  $\Xi_1$ ,  $\oplus C$ , and  $\mathbf{B}_{\mathbf{c}(\Omega, X)}$  in  $\mathbf{c}(\Omega, X)$  respectively are  $\Sigma_0$ ,  $\Sigma_1$ ,  $\oplus C^\odot$  and  $\mathbf{B}_{\ell^1(\Omega, X')}$  in  $\ell^1(\Omega, X')$ . Since  $\alpha \mathbf{B}_{\ell^1(\Omega, X')}$  is  $\sigma(\ell^1(\Omega, X'), \mathbf{c}(\Omega, X))$ -compact (by The Banach-Alaoglu Theorem [6, Theorem V.3.1]) and the sets  $\oplus C^\odot$  and  $\Sigma_0$  are  $\sigma(\ell^1(\Omega, X'), \mathbf{c}(\Omega, X))$ -closed,  $\alpha \mathbf{B}_{\ell^1(\Omega, X')} \cap \oplus C^\odot + \Sigma_0$  is  $\sigma(\ell^1(\Omega, X'), \mathbf{c}(\Omega, X))$ -closed. The result now follows from Theorem 3.4(1)(b).

We prove (2). By Lemma 4.4, the one-sided polars of the sets  $\Sigma_0$ ,  $\Sigma_1$ ,  $\oplus C$  and  $\mathbf{B}_{\ell^1(\Omega, X)}$  in  $\ell^1(\Omega, X)$  respectively are  $\Xi_\infty$ ,  $\Xi_1$ ,  $\oplus C^\odot$  and  $\mathbf{B}_{\ell^\infty(\Omega, X')}$  in  $\ell^\infty(\Omega, X')$ . By Theorem 3.4(2)(a), if  $\ell^1(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^1(\Omega, X)})$  in  $\ell^1(\Omega, X)$  for all  $\beta > \alpha$ , then the space  $\ell^\infty(\Omega, X')$  is normal with respect to  $(\oplus C^\odot, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^\infty(\Omega, X')})$  in  $\ell^\infty(\Omega, X')$ .

Conversely, let  $\ell^\infty(\Omega, X')$  be normal with respect to  $(\oplus C^\odot, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^\infty(\Omega, X')})$  in  $\ell^\infty(\Omega, X')$ . Invoking Lemma 2.3, it can be seen that  $\alpha \mathbf{B}_{\ell^1(\Omega, X)} \cap \oplus C + \Sigma_0$  is  $\sigma(\ell^1(\Omega, X), \ell^\infty(\Omega, X'))$ -cs-closed. Furthermore, the  $\sigma(\ell^1(\Omega, X), \ell^\infty(\Omega, X'))$ -cs-compact set  $\mathbf{B}_{\ell^1(\Omega, X)}$  is contained in  $\Sigma_1$ . By Lemma 3.3,

$$\Sigma_1 \subseteq \overline{(\alpha \mathbf{B}_{\ell^1(\Omega, X)} \cap \oplus C + \Sigma_0)^{\sigma(\ell^1(\Omega, X), \ell^\infty(\Omega, X'))}},$$

and since the norm-closure and  $\sigma(\ell^1(\Omega, X), \ell^\infty(\Omega, X'))$ -closure of  $\alpha \mathbf{B}_{\ell^1(\Omega, X)} \cap \oplus C + \Sigma_0$  coincide (cf. [6, Theorem V.1.4]), it holds that, for every  $r > 0$  and  $\xi \in \Sigma_1$ ,  $(\xi - r \mathbf{B}_{\ell^1(\Omega, X)}) \cap (\alpha \mathbf{B}_{\ell^1(\Omega, X)} \cap \oplus C + \Sigma_0) \neq \emptyset$ . Finally, by Theorem 3.4(2)(b),  $\ell^1(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^1(\Omega, X)})$  in  $\ell^1(\Omega, X)$  for every  $\beta > \alpha$ .

The assertions (3) and (4) follow similarly:

We prove (3). By Lemma 4.4, the one-sided polars of the sets  $\Xi_\infty$ ,  $\Xi_1$ ,  $\oplus C$ , and  $\mathbf{B}_{\ell^p(\Omega, X)}$  in  $\ell^p(\Omega, X)$  respectively are  $\Sigma_0$ ,  $\Sigma_1$ ,  $\oplus C^\odot$  and  $\mathbf{B}_{\ell^q(\Omega, X')}$  in  $\ell^q(\Omega, X')$ . As in (1),  $\alpha \mathbf{B}_{\ell^q(\Omega, X')} \cap \oplus C^\odot + \Sigma_0$  is  $\sigma(\ell^q(\Omega, X'), \ell^p(\Omega, X))$ -closed. The result now follows from Theorem 3.4(1)(b).



We prove (4). By Theorem 3.4(2)(a), if  $\ell^p(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^p(\Omega, X)})$  in  $\ell^p(\Omega, X)$  for all  $\beta > \alpha$ , then  $\ell^q(\Omega, X')$  is normal with respect to  $(\oplus C^\circ, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^q(\Omega, X')})$  in  $\ell^q(\Omega, X')$ .

Conversely, let  $\ell^q(\Omega, X')$  be normal with respect to  $(\oplus C^\circ, \Xi_\infty, \alpha \Xi_1, \mathbf{B}_{\ell^q(\Omega, X')})$  in  $\ell^q(\Omega, X')$ . Invoking Lemma 2.3, it can be seen that  $\alpha \mathbf{B}_{\ell^p(\Omega, X)} \cap \oplus C + \Sigma_0$  is  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -cs-closed. Also, since  $|\Omega|$  is finite, by the Hölder- and Minkowski inequalities, the  $\|\cdot\|_1$ - and  $\|\cdot\|_p$ -norms on  $\ell^p(\Omega, X)$  are equivalent. By Lemma 2.3,  $\mathbf{B}_{\ell^1(\Omega, X)}$  is a  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -pre-cs-compact  $\|\cdot\|_p$ -neighborhood of zero contained in  $\Sigma_1$  (being a subset of the cs-compact set  $\gamma \mathbf{B}_{\ell^p(\Omega, X)}$  for some  $\gamma > 0$ ). By Lemma 3.3,

$$\Sigma_1 \subseteq \overline{(\alpha \mathbf{B}_{\ell^p(\Omega, X)} \cap \oplus C + \Sigma_0)^{\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))}},$$

and since the norm-closure and  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -closure of  $\alpha \mathbf{B}_{\ell^p(\Omega, X)} \cap \oplus C + \Sigma_0$  coincide (cf. [6, Theorem V.1.4]), it holds that, for every  $r > 0$  and  $\xi \in \Sigma_1$ ,  $(\xi - r \mathbf{B}_{\ell^1(\Omega, X)}) \cap (\alpha \mathbf{B}_{\ell^p(\Omega, X)} \cap \oplus C + \Sigma_0) \neq \emptyset$  (since  $\mathbf{B}_{\ell^1(\Omega, X)}$  is a  $\|\cdot\|_p$ -norm-neighborhood of zero). Finally, by Theorem 3.4(2)(b),  $\ell^p(\Omega, X)$  is conormal with respect to  $(\oplus C, \Sigma_0, \Sigma_1, \beta \mathbf{B}_{\ell^p(\Omega, X)})$  in  $\ell^p(\Omega, X)$  for every  $\beta > \alpha$ .  $\square$

With a straightforward calculation, which we omit, the above theorem can be reformulated into the following corollary.

**Corollary 4.7.** *Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$  and  $\alpha \geq 1$ .*

- (1) *The following are equivalent:*
  - (a) *If  $\xi \in \mathbf{c}(\Omega, X)$  and  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ , then  $\|x\| \leq \alpha \|\xi\|_\infty$ .*
  - (b) *For every  $\phi \in X'$ , there exists an element  $\eta \in \oplus C^\circ \subseteq \ell^1(\Omega, X')$  such that  $\phi = \Sigma \eta$  and  $\|\eta\|_1 \leq \alpha \|\phi\|$ .*
- (2) *The following are equivalent:*
  - (a) *For any  $x \in X$  and  $\beta > \alpha$ , there exists an element  $\xi \in \oplus C \subseteq \ell^1(\Omega, X)$  such that  $x = \Sigma \xi$  and  $\|\xi\|_1 \leq \beta \|x\|$ .*
  - (b) *If  $\eta \in \ell^\infty(\Omega, X')$  and  $\phi \in \bigcap_{\omega \in \Omega} (\eta + C_\omega^\circ)$ , then  $\|\phi\| \leq \alpha \|\eta\|_\infty$ .*

*If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:*

- (3) *The following are equivalent:*
  - (a) *If  $\xi \in \ell^p(\Omega, X)$  and  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ , then  $\|x\| \leq \alpha \|\xi\|_p$ .*
  - (b) *For every  $\phi \in X'$ , there exists an element  $\eta \in \oplus C^\circ \subseteq \ell^q(\Omega, X')$  such that  $\phi = \Sigma \eta$  and  $\|\eta\|_q \leq \alpha \|\phi\|$ .*
- (4) *The following are equivalent:*
  - (a) *For every  $x \in X$  and  $\beta > \alpha$ , there exists an element  $\xi \in \oplus C \subseteq \ell^p(\Omega, X)$  such that  $x = \Sigma \xi$  and  $\|\xi\|_p \leq \beta \|x\|$ .*
  - (b) *If  $\eta \in \ell^q(\Omega, X')$  and  $\phi \in \bigcap_{\omega \in \Omega} (\eta + C_\omega^\circ)$ , then  $\|\phi\| \leq \alpha \|\eta\|_q$ .*

A similar argument as was employed in Theorem 4.6 will also establish the following theorem.

**Theorem 4.8.** *Let  $\alpha \geq 1$ . Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$ .*

- (1) *The space  $\ell^1(\Omega, X)$  is additive with respect to  $(\oplus C, \{0\}, \alpha \mathbf{B}_{\ell^1(\Omega, X)}, \Sigma_1)$  in  $\ell^1(\Omega, X)$  if and only if the space  $\ell^\infty(\Omega, X')$  is coadditive with respect to  $(\oplus C^\circ, \ell^\infty(\Omega, X'), \mathbf{B}_{\ell^\infty(\Omega, X')}, \alpha \Xi_1)$  in  $\ell^\infty(\Omega, X')$ .*
- (2) *The space  $\mathbf{c}(\Omega, X)$  is coadditive with respect to  $(\oplus C, \mathbf{c}(\Omega, X), \mathbf{B}_{\mathbf{c}(\Omega, X)}, \beta \Xi_1)$  in  $\mathbf{c}(\Omega, X)$  for every  $\beta > \alpha$  if and only if the space  $\ell^1(\Omega, X')$  is additive with respect to  $(\oplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^1(\Omega, X')}, \Sigma_1)$  in  $\ell^1(\Omega, X')$ .*

*If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:*

- (3) *The space  $\ell^p(\Omega, X)$  is additive with respect to  $(\oplus C, \{0\}, \alpha \mathbf{B}_{\ell^p(\Omega, X)}, \Sigma_1)$  in  $\ell^p(\Omega, X)$  if and only if the space  $\ell^q(\Omega, X')$  is coadditive with respect to  $(\oplus C^\circ, \ell^q(\Omega, X'), \mathbf{B}_{\ell^q(\Omega, X')}, \alpha \Xi_1)$  in  $\ell^q(\Omega, X')$ .*
- (4) *The space  $\ell^p(\Omega, X)$  is coadditive with respect to  $(\oplus C, \ell^p(\Omega, X), \mathbf{B}_{\ell^p(\Omega, X)}, \beta \Xi_1)$  in  $\ell^p(\Omega, X)$  for all  $\beta > \alpha$  if and only if the space  $\ell^q(\Omega, X')$  is additive with respect to  $(\oplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^q(\Omega, X')}, \Sigma_1)$  in  $\ell^q(\Omega, X')$ .*

*Proof.* We prove (1). By Lemma 4.4, the one-sided polars of the sets  $\Sigma_1$ ,  $\oplus C$  and  $\mathbf{B}_{\ell^1(\Omega, X)}$  in  $\ell^1(\Omega, X)$  respectively equal  $\Xi_1$ ,  $\oplus C^\circ$ , and  $\mathbf{B}_{\ell^\infty(\Omega, X')}$  in  $\ell^\infty(\Omega, X')$ . We notice that  $\alpha \Xi_1$  in  $\ell^\infty(\Omega, X')$  is  $\sigma(\ell^\infty(\Omega, X'), \ell^1(\Omega, X))$ -compact, since it is  $\sigma(\ell^\infty(\Omega, X'), \ell^1(\Omega, X))$ -closed in  $\mathbf{B}_{\ell^\infty(\Omega, X')}$ , and hence, the set  $\alpha \Xi_1 + \oplus C^\circ$  is  $\sigma(\ell^\infty(\Omega, X'), \ell^1(\Omega, X))$ -closed. By Theorem 3.5(1)(b) the result follows.

We prove (2). By Lemma 4.4, the one-sided polars of the sets  $\Xi_1$ ,  $\oplus C$ , and  $\mathbf{B}_{\mathbf{c}(\Omega, X)}$  in  $\mathbf{c}(\Omega, X)$  respectively are  $\Sigma_1$ ,  $\oplus C^\circ$  and  $\mathbf{B}_{\ell^1(\Omega, X')}$  in  $\ell^1(\Omega, X')$ . From Theorem 3.5(2)(a) we immediately conclude that, if  $\mathbf{c}(\Omega, X)$  is coadditive with respect to  $(\oplus C, \mathbf{c}(\Omega, X), \mathbf{B}_{\mathbf{c}(\Omega, X)}, \beta \Xi_1)$  in  $\mathbf{c}(\Omega, X)$  for every  $\beta > \alpha$ , then  $\ell^1(\Omega, X')$  is additive with respect to  $(\oplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^1(\Omega, X')}, \Sigma_1)$  in  $\ell^1(\Omega, X')$ .

Conversely, let  $\ell^1(\Omega, X')$  be additive with respect to  $(\oplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^1(\Omega, X')}, \Sigma_1)$  in  $\ell^1(\Omega, X')$ . Invoking Lemma 2.3, it can be seen that the set  $\alpha \Xi_1 + \oplus C$  is  $\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))$ -cs-closed ( $\alpha \Xi_1 + \oplus C$  is  $\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))$ -closed and convex). By Lemma 3.3,

$$\mathbf{B}_{\mathbf{c}(\Omega, X)} \subseteq \overline{(\alpha \Xi_1 + \oplus C)^{\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))}} \cap \mathbf{c}(\Omega, X) = \overline{(\alpha \Xi_1 + \oplus C)^{\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))}}.$$

Since the norm-closure and  $\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))$ -closure of  $\alpha \Xi_1 + \oplus C$  coincide (cf. [6, Theorem V.1.4]), we have, for every  $r > 0$  and  $b \in \mathbf{B}_{\mathbf{c}(\Omega, X)}$ , that  $(b - r \mathbf{B}_{\mathbf{c}(\Omega, X)}) \cap (\alpha \Xi_1 + \oplus C) \neq \emptyset$ . But  $\mathbf{B}_{\mathbf{c}(\Omega, X)}$  is a  $\sigma(\mathbf{c}(\Omega, X), \ell^1(\Omega, X'))$ -cs-compact set, so by Theorem 3.5(2)(b),  $\mathbf{c}(\Omega, X)$  is coadditive with respect to  $(\oplus C, \mathbf{c}(\Omega, X), \mathbf{B}_{\mathbf{c}(\Omega, X)}, \beta \Xi_1)$  in  $\mathbf{c}(\Omega, X)$  for every  $\beta > \alpha$ .

The assertions (3) and (4) follow similarly:

We prove (3). By Lemma 4.4, the one-sided polars of the sets  $\Sigma_1$ ,  $\oplus C$  and  $\mathbf{B}_{\ell^p(\Omega, X)}$  in  $\ell^p(\Omega, X)$  respectively equal  $\Xi_1$ ,  $\oplus C^\circ$ , and  $\mathbf{B}_{\ell^q(\Omega, X')}$  in  $\ell^q(\Omega, X')$ . We notice  $\alpha \Xi_1 + \oplus C^\circ$  is  $\sigma(\ell^q(\Omega, X'), \ell^p(\Omega, X))$ -closed. By Theorem 3.5(1)(b) the result follows.

We prove (4). By Lemma 4.4, the one-sided polars of the sets  $\Xi_1$ ,  $\bigoplus C$ , and  $\mathbf{B}_{\ell^p(\Omega, X)}$  in  $\ell^p(\Omega, X)$  respectively are  $\Sigma_1$ ,  $\bigoplus C^\circ$  and  $\mathbf{B}_{\ell^q(\Omega, X')}$  in  $\ell^q(\Omega, X')$ . From Theorem 3.5(2)(a) we immediately conclude that, if  $\ell^p(\Omega, X)$  is coadditive with respect to  $(\bigoplus C, \ell^p(\Omega, X), \mathbf{B}_{\ell^p(\Omega, X)}, \beta \Xi_1)$  in  $\ell^p(\Omega, X)$  for every  $\beta > \alpha$ , then  $\ell^q(\Omega, X')$  is additive with respect to  $(\bigoplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^q(\Omega, X')}, \Sigma_1)$  in  $\ell^q(\Omega, X')$ .

Conversely, let  $\ell^q(\Omega, X')$  be additive with respect to  $(\bigoplus C^\circ, \{0\}, \alpha \mathbf{B}_{\ell^q(\Omega, X')}, \Sigma_1)$  in  $\ell^q(\Omega, X')$ . Invoking Lemma 2.3, it can be seen that the set  $\alpha \Xi_1 + \bigoplus C$  is  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -cs-closed. By Lemma 3.3,

$$\mathbf{B}_{\ell^p(\Omega, X)} \subseteq \overline{(\alpha \Xi_1 + \bigoplus C)^{\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))}} \cap \ell^p(\Omega, X) = \overline{(\alpha \Xi_1 + \bigoplus C)^{\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))}}.$$

Since the norm-closure and  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -closure of  $\alpha \Xi_1 + \bigoplus C$  coincide (cf. [6, Theorem V.1.4]), we have, for every  $r > 0$  and  $b \in \mathbf{B}_{\ell^p(\Omega, X)}$ , that  $(b - r \mathbf{B}_{\ell^p(\Omega, X)}) \cap (\alpha \Xi_1 + \bigoplus C) \neq \emptyset$ . But  $\mathbf{B}_{\ell^p(\Omega, X)}$  is a  $\sigma(\ell^p(\Omega, X), \ell^q(\Omega, X'))$ -cs-compact set, so by Theorem 3.5(2)(b),  $\ell^p(\Omega, X)$  is coadditive with respect to  $(\bigoplus C, \ell^p(\Omega, X), \mathbf{B}_{\ell^p(\Omega, X)}, \beta \Xi_1)$  in  $\ell^p(\Omega, X)$  for every  $\beta > \alpha$ .  $\square$

Again, a straightforward calculation which we omit, allows the reformulation of the above theorem into the following corollary.

**Corollary 4.9.** *Let  $X$  be a real Banach space and  $\{C_\omega\}_{\omega \in \Omega}$  a collection of closed cones in  $X$  and  $\alpha \geq 1$ .*

- (1) *The following are equivalent:*
  - (a) *If  $\xi \in \bigoplus C \subseteq \ell^1(\Omega, X)$ , then  $\|\xi\|_1 \leq \alpha \|\Sigma \xi\|$ .*
  - (b) *For every  $\eta \in \ell^\infty(\Omega, X')$ , there exists some  $\phi \in \bigcap_{\omega \in \Omega} (\eta_\omega - C_\omega^\circ)$  with  $\|\phi\| \leq \alpha \|\eta\|_\infty$ .*
- (2) *The following are equivalent:*
  - (a) *For every  $\xi \in \mathbf{c}(\Omega, X)$  and  $\beta > \alpha$ , there exists some  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega - C_\omega)$  with  $\|x\| \leq \beta \|\xi\|_\infty$ .*
  - (b) *If  $\eta \in \bigoplus C^\circ \subseteq \ell^1(\Omega, X')$ , then  $\|\eta\|_1 \leq \alpha \|\Sigma \eta\|$ .*

*If, in addition,  $|\Omega| < \infty$  and  $1 \leq p, q \leq \infty$ , with  $p^{-1} + q^{-1} = 1$ , then:*

- (3) *The following are equivalent:*
  - (a) *If  $\xi \in \bigoplus C \subseteq \ell^p(\Omega, X)$ , then  $\|\xi\|_p \leq \alpha \|\Sigma \xi\|$ .*
  - (b) *For every  $\eta \in \ell^q(\Omega, X')$ , there exists some  $\phi \in \bigcap_{\omega \in \Omega} (\eta_\omega - C_\omega^\circ)$  with  $\|\phi\| \leq \alpha \|\eta\|_q$ .*
- (4) *The following are equivalent:*
  - (a) *For every  $\xi \in \ell^p(\Omega, X)$  and  $\beta > \alpha$ , there exists some  $x \in \bigcap_{\omega \in \Omega} (\xi_\omega - C_\omega)$  with  $\|x\| \leq \beta \|\xi\|_p$ .*
  - (b) *If  $\eta \in \bigoplus C^\circ \subseteq \ell^q(\Omega, X')$ , then  $\|\eta\|_q \leq \alpha \|\Sigma \eta\|$ .*

## 5. APPLICATION: GEOMETRIC PROPERTIES OF CONES IN $C^*$ -ALGEBRAS

In this section we give an elementary application of geometric duality theory to naturally occurring cones in  $C^*$ -algebras.

Throughout this section  $A$  will be a  $C^*$ -algebra and  $A'$  its dual. We will view both  $A$  and  $A'$  as vector spaces over  $\mathbb{R}$ , and define the (real) bilinear map  $\langle \cdot | \cdot \rangle : A \times A' \rightarrow \mathbb{R}$  by  $\langle a | \phi \rangle := \operatorname{Re} \phi(a)$  ( $a \in A$ ,  $\phi \in A'$ ). By The Separation Theorem [18, Theorem 3.21],  $\langle \cdot | \cdot \rangle$  becomes a duality as defined in Section 2.1. As usual, we define the closed cone of positive elements in  $A$  by  $A_+ := \{a^*a \in A \mid a \in A\}$  and the closed

cone of positive functionals in  $A'$  by  $A'_+ := \{\phi \in A' \mid \phi(a^*a) \geq 0 \ \forall a \in A\}$ . We stress, since  $A$  is a complex space, that  $A'_+ \neq -A'_+{}^\circ = \{\phi \in A' \mid \operatorname{Re} \phi(a^*a) \geq 0, \ \forall a \in A\}$ . For  $a, b \in A$  and  $\phi, \varphi \in A'$ , by  $a \leq b$  and  $\phi \leq \varphi$  we respectively mean  $b \in a + A_+$  and  $\varphi \in \phi + A'_+$ . We define the real subspaces of self-adjoint elements in  $A$  by  $A_{\text{sa}} := \{\phi \in A \mid a = a^*\}$  and of self-adjoint functionals on  $A$  by  $A'_{\text{sa}} := \{\phi \in A' \mid \phi(a^*) = \overline{\phi(a)}, \ \forall a \in A\}$ .

**Lemma 5.1.** *Let  $A$  be a  $C^*$ -algebra and  $A'$  its dual, and the duality  $\langle \cdot, \cdot \rangle : A \times A' \rightarrow \mathbb{R}$  as defined above. Then,*

- (1)  $A_+^\circ = -A'_+ + iA'_{\text{sa}}$ .
- (2)  $(iA_+)^\circ = -iA'_+ + A'_{\text{sa}}$ .
- (3)  $(A'_+)^\circ = -A_+ + iA_{\text{sa}}$ .
- (4)  $(iA'_+)^\circ = -iA_+ + A_{\text{sa}}$ .

*Proof.* We prove (1). For  $\phi \in -A'_+$ ,  $\varphi \in A'_{\text{sa}}$  and  $a \in A$ , since  $\varphi(a^*a) \in \mathbb{R}$ , it is clear that  $\operatorname{Re}(\phi + i\varphi)(a^*a) = \phi(a^*a) \leq 0$ . Hence  $-A'_+ + iA'_{\text{sa}} \subseteq A_+^\circ$ . Conversely, let  $\phi \in A_+^\circ$ . Defining  $\phi^* \in A'$  by  $\phi^*(a) := \overline{\phi(a^*)}$  ( $a \in A$ ), and  $\varphi, \psi \in A$  by  $\varphi := 2^{-1}(\phi + \phi^*)$  and  $\psi := (2i)^{-1}(\phi - \phi^*)$ , so that  $\varphi, \psi \in A'_{\text{sa}}$  and  $\phi = \varphi + i\psi$ . Hence, for all  $a \in A$ ,  $\varphi(a^*a) = \operatorname{Re} \phi(a^*a) = \langle a^*a, \phi \rangle \leq 0$ , so that  $\varphi \in -A'_+$  and  $A_+^\circ \subseteq -A'_+ + iA'_{\text{sa}}$ . A similar argument will establish (2).

We prove (3). Let  $a \in -A_+$  and  $b \in A_{\text{sa}}$ . Then for  $\phi \in A'_+$ ,  $\langle a + ib, \phi \rangle = \operatorname{Re}(\phi(a) + i\phi(b)) = \phi(a) \leq 0$ , so that  $a + ib \in A_+^\circ$ . Conversely, let  $a \in A_+^\circ$ . We write  $a = 2^{-1}(a + a^*) + i(2i)^{-1}(a - a^*)$ . Then, for all  $\phi \in A'_+$ , we have  $0 \geq \langle a, \phi \rangle = \operatorname{Re} \phi(a) = 2^{-1}\phi(a + a^*)$ . Therefore,  $2^{-1}(a + a^*) \in -A_+$  (by [9, Proposition 2.6.2]), and hence  $a = 2^{-1}(a + a^*) + i(2i)^{-1}(a - a^*) \in -A_+ + iA_{\text{sa}}$ . A similar argument will establish (4).  $\square$

The following result, originally due to Grothendieck ([12] via [17, Theorem 3.2.5]), shows that the cones  $\{A'_+, -A'_+, iA'_+, -iA'_+\}$  generate the dual of a  $C^*$ -algebra  $A$ .

**Theorem 5.2.** *Let  $A$  be a  $C^*$ -algebra. If  $\phi \in A'_{\text{sa}}$ , then there exist unique  $\phi_+, \phi_- \in A'_+$  with  $\phi = \phi_+ - \phi_-$  and  $\|\phi\| = \|\phi_+\| + \|\phi_-\|$ .*

We can now apply our results from the previous sections to obtain the following geometric properties of naturally occurring cones in  $C^*$ -algebras and their duals. The results presented here are stronger than what is usually presented in the canon (e.g., [9, 1.6.9]).

**Theorem 5.3.** *Let  $A$  be a  $C^*$ -algebra and  $A'$  its dual.*

- (1) *For  $b_1, b_2, b_3, b_4 \in A$ , if*

$$a \in (b_1 + A_+ + iA_{\text{sa}}) \cap (b_2 - A_+ + iA_{\text{sa}}) \\ \cap (b_3 + iA_+ + A_{\text{sa}}) \cap (b_4 - iA_+ + A_{\text{sa}})$$

$$\text{then } \|a\| \leq \max\{\|b_1\|, \|b_2\|\} + \max\{\|b_3\|, \|b_4\|\}.$$

- (2) *For  $b_1, b_2, b_3, b_4 \in A$ , if*

$$a \in (b_1 + A_+) \cap (b_2 - A_+) \cap (b_3 + iA_+) \cap (b_4 - iA_+),$$

$$\text{then } \|a\| \leq \max\{\|b_1\|, \|b_2\|\} + \max\{\|b_3\|, \|b_4\|\}.$$

- (3) *For  $a, b, c \in A$ , if  $a \leq b \leq c$ , then  $\|b\| \leq 2 \max\{\|a\|, \|c\|\}$ .*
- (4) *For  $a, b, c \in A_{\text{sa}}$ , if  $a \leq b \leq c$ , then  $\|b\| \leq \max\{\|a\|, \|c\|\}$ .*

(5) For  $\phi_1, \phi_2, \phi_3, \phi_4 \in A'$ , if

$$\begin{aligned} \varphi \in & (\phi_1 + A'_+ + iA'_{\text{sa}}) \cap (\phi_2 - A'_+ + iA'_{\text{sa}}) \\ & \cap (\phi_3 + iA'_+ + A'_{\text{sa}}) \cap (\phi_4 - iA'_+ + A'_{\text{sa}}) \end{aligned}$$

then  $\|\varphi\| \leq \sum_{j=1}^4 \|\phi_j\|$ .

(6) For  $\phi_1, \phi_2, \phi_3, \phi_4 \in A'$ , if

$$\varphi \in (\phi_1 + A'_+) \cap (\phi_2 - A'_+) \cap (\phi_3 + iA'_+) \cap (\phi_4 - iA'_+),$$

then  $\|\varphi\| \leq \sum_{j=1}^4 \|\phi_j\|$ .

(7) For  $\rho, \phi, \psi \in A'$ , if  $\rho \leq \phi \leq \psi$ , then  $\|\phi\| \leq 2(\|\rho\| + \|\psi\|)$ .

(8) For  $\rho, \phi, \psi \in A'_{\text{sa}}$ , if  $\rho \leq \phi \leq \psi$ , then  $\|\phi\| \leq \|\rho\| + \|\psi\|$ .

*Proof.* We prove (1) by applying Theorem 3.4. We define the Banach spaces  $Y := (A \oplus_\infty A) \oplus_1 (A \oplus_\infty A)$  and  $Z := (A' \oplus_1 A') \oplus_\infty (A' \oplus_1 A')$ , and define the duality  $\langle \cdot, \cdot \rangle : Y \times Z \rightarrow \mathbb{R}$  by

$$\langle (a_1, a_2, a_3, a_4), (\phi_1, \phi_2, \phi_3, \phi_4) \rangle := \sum_{j=1}^4 \langle a_j, \phi_j \rangle = \sum_{j=1}^4 \text{Re } \phi_j(a_j),$$

where  $(a_1, a_2, a_3, a_4) \in Y$  and  $(\phi_1, \phi_2, \phi_3, \phi_4) \in Z$ . We define the sets

$$\begin{aligned} C &:= \{ (a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}) \in Y \mid a_{j,k} \in (-1)^k (i)^j A_+ + (i)^{1-j} A_{\text{sa}} \}, \\ \Xi_\infty &:= \{ (a, a, a, a) \in Y \mid a \in A \}, \\ \Xi_1 &:= \{ (a, a, a, a) \in Y \mid a \in A, \|a\| \leq 1 \}, \\ E &:= \{ (\phi_{0,0}, \phi_{0,1}, \phi_{1,0}, \phi_{1,1}) \in Z \mid \phi_{j,k} \in (-1)^{1-k} (i)^j A'_+ \}, \\ \Sigma_0 &:= \left\{ (\phi_1, \phi_2, \phi_3, \phi_4) \in Z \mid \sum_{j=1}^4 \phi_j = 0 \right\}, \\ \Sigma_1 &:= \left\{ (\phi_1, \phi_2, \phi_3, \phi_4) \in Z \mid \left\| \sum_{j=1}^4 \phi_j \right\| \leq 1 \right\}. \end{aligned}$$

An easy computation together with Lemma 5.1 will show that  $C^\circ = E$ ,  $\Xi_\infty^\circ = \Sigma_0$ ,  $\Xi_1^\circ = \Sigma_1$  and  $\mathbf{B}_Y^\circ = \mathbf{B}_Z$ . By Theorem 5.2, for every  $\zeta \in \Sigma_1$ , there exist  $\phi_1, \phi_2, \phi_3, \phi_4 \in A'_+$  with  $\sum_{j=1}^4 \zeta_j = \phi_2 - \phi_1 + i(\phi_4 - \phi_3)$  and

$$\max \{ \|\phi_1\| + \|\phi_2\|, \|\phi_4\| + \|\phi_3\| \} \leq \left\| \sum_{j=1}^4 \zeta_j \right\| \leq 1,$$

i.e.,  $\Phi := (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbf{B}_Z \cap E$ . Therefore,  $\zeta = \Phi + (\zeta - \Phi) \in \mathbf{B}_Z \cap E + \Sigma_0$  and we conclude that  $\Sigma_1 \subseteq \mathbf{B}_Z \cap E + \Sigma_0$ , i.e.,  $Z$  is conormal with respect to  $(E, \Sigma_0, \Sigma_1, \mathbf{B}_Z)$ . Since  $\mathbf{B}_Z$  is  $\sigma(Z, Y)$ -compact (by The Banach-Alaoglu Theorem [6, Theorem V.3.1]) and  $\Sigma_0$  is  $\sigma(Z, Y)$ -closed,  $\mathbf{B}_Z \cap E + \Sigma_0$  is  $\sigma(Z, Y)$ -closed. Hence, by Theorem 3.4(1),  $Y$  is normal with respect to  $(C, \Xi_\infty, \Xi_1, \mathbf{B}_Y)$ , i.e.,  $(\mathbf{B}_Y + C) \cap \Xi_\infty \subseteq \Xi_1$ . Therefore, for  $b_1, b_2, b_3, b_4 \in A$ , if

$$a \in (b_1 + A_+ + iA_{\text{sa}}) \cap (b_2 - A_+ + iA_{\text{sa}}) \cap (b_3 + iA_+ + A_{\text{sa}}) \cap (b_4 - iA_+ + A_{\text{sa}}),$$

then  $\|a\| \leq \max \{ \|b_1\|, \|b_2\| \} + \max \{ \|b_3\|, \|b_4\| \}$ .

We prove (5). For every  $a \in A$ , there exist elements  $a_1, a_2, a_3, a_4 \in A_+$  with  $a = a_1 - a_2 + ia_3 - ia_4$  and  $\max_{j \in \{1,2,3,4\}} \|a_j\| \leq \|a\|$ . By Lemma 5.1 and Corollary 4.7(4), for  $\phi_{j,k} \in A'$  ( $j, k \in \{0,1\}$ ), if  $\varphi \in \bigcap_{j,k \in \{0,1\}} (\phi_{j,k} + (-1)^j (i)^k A'_+ + (i)^{1-k} A'_{\text{sa}})$ , then  $\|\varphi\| \leq \sum_{j,k \in \{0,1\}} \|\phi_{j,k}\|$ .

The assertions (2) and (6) follow immediately from (1) and (5) respectively, since  $A_+ \subseteq A_+ + A_{sa}$  and  $A'_+ \subseteq A'_+ + A'_{sa}$ .

We prove (3). If  $a, b, c \in A$  satisfy  $a \leq b \leq c$ , then

$$\begin{aligned} b &\in (a + A_+) \cap (c - A_+) \cap (a + A_{sa}) \cap (c + A_{sa}) \\ &\subseteq (a + A_+ + iA_{sa}) \cap (c - A_+ + iA_{sa}) \cap (a + iA_+ + A_{sa}) \cap (c - iA_+ + A_{sa}). \end{aligned}$$

Hence, by (1), we obtain  $b \leq 2 \max \{ \|a\|, \|c\| \}$ .

We prove (4). If  $a, b, c \in A_{sa}$  satisfy  $a \leq b \leq c$ , then

$$\begin{aligned} b &\in (a + A_+) \cap (c - A_+) \cap (0 + A_{sa}) \cap (0 + A_{sa}) \\ &\subseteq (a + A_+ + iA_{sa}) \cap (c - A_+ + iA_{sa}) \cap (0 + iA_+ + A_{sa}) \cap (0 - iA_+ + A_{sa}). \end{aligned}$$

Hence, by (1), we obtain  $b \leq \max \{ \|a\|, \|c\| \}$ .

We prove (7). If  $\rho, \phi, \psi \in A'$  satisfy  $\rho \leq \phi \leq \psi$ , then

$$\begin{aligned} \phi &\in (\rho + A'_+) \cap (\psi - A'_+) \cap (\rho + A'_{sa}) \cap (\psi + A'_{sa}) \\ &\subseteq (\rho + A'_+ + iA'_{sa}) \cap (\psi - A'_+ + iA'_{sa}) \cap (\rho + iA'_+ + A'_{sa}) \cap (\psi - iA'_+ + A'_{sa}), \end{aligned}$$

so, by (5),  $\|\phi\| \leq 2(\|\rho\| + \|\psi\|)$ .

We prove (8). If  $\rho, \phi, \psi \in A'_{sa}$  satisfy  $\rho \leq \phi \leq \psi$ , then

$$\begin{aligned} \phi &\in (\rho + A'_+) \cap (\psi - A'_+) \cap (0 + A'_{sa}) \cap (0 + A'_{sa}) \\ &\subseteq (\rho + A'_+ + iA'_{sa}) \cap (\psi - A'_+ + iA'_{sa}) \cap (0 + iA'_+ + A'_{sa}) \cap (0 - iA'_+ + A'_{sa}), \end{aligned}$$

so, by (5),  $\|\phi\| \leq \|\rho\| + \|\psi\|$ .  $\square$

*Remark 5.4.* Some of the above results are known: The earliest references to (4) and (8) known to the author is [5, Examples 1.1.7 and 1.2.5]. Particularly, (4) can be established through an elementary application of Grothendieck's 1957 result Theorem 5.2, and Grosberg and Krein's 1939 result Theorem 1.2(1) from the introduction, so is (at least in theory) quite old. No references to (1)–(3) and (5)–(7) are known to the author.

## REFERENCES

1. C.D. Aliprantis and K.C. Border, *Infinite dimensional analysis*, third ed., Springer, Berlin, 2006.
2. T. Andô, *On fundamental properties of a Banach space with a cone*, Pacific J. Math. **12** (1962), 1163–1169.
3. L. Asimow, *Directed Banach spaces of affine functions*, Trans. Amer. Math. Soc. **143** (1969), 117–132.
4. L. Asimow and A.J. Ellis, *Convexity theory and its applications in functional analysis*, vol. 16, Academic Press, Inc., London-New York, 1980.
5. C.J.K. Batty and D.W. Robinson, *Positive one-parameter semigroups on ordered Banach spaces*, Acta Appl. Math. **2** (1984), no. 3-4, 221–296.
6. J.B. Conway, *A course in functional analysis*, second ed., Springer-Verlag, New York, 1990.
7. M. de Jeu and M. Messerschmidt, *A strong open mapping theorem for surjections from cones onto Banach spaces*, Adv. Math. **259** (2014), 43–66.
8. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge University Press, Cambridge, 1995.
9. J. Dixmier, *C\*-algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

10. A.J. Ellis, *The duality of partially ordered normed linear spaces*, J. London Math. Soc. **39** (1964), 730–744.
11. J. Grosberg and M. Krein, *Sur la décomposition des fonctionnelles en composantes positives*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **25** (1939), 723–726.
12. A. Grothendieck, *Un résultat sur le dual d'une  $C^*$ -algèbre*, J. Math. Pures Appl. (9) **36** (1957), 97–108.
13. G.J.O. Jameson, *Ordered linear spaces*, Springer-Verlag, Berlin-New York, 1970.
14. ———, *Convex series*, Proc. Cambridge Philos. Soc. **72** (1972), 37–47.
15. M. Krein, *Propriétés fondamentales des ensembles coniques normaux dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **28** (1940), 13–17.
16. M. Messerschmidt, *Normality of spaces of operators, and quasi-lattices*, arXiv preprint arXiv:1307.1415 (2013).
17. G.K. Pedersen,  *$C^*$ -algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press, Inc., London-New York, 1979.
18. W. Rudin, *Functional analysis*, second ed., McGraw-Hill Inc., New York, 1991.
19. B. Walsh, *Ordered vector sequence spaces and related classes of linear operators*, Math. Ann. **206** (1973), 89–138.
20. A.W. Wickstead, *Compact subsets of partially ordered Banach spaces*, Math. Ann. **212** (1975), 271–284.
21. Y.C. Wong and K.F. Ng, *Partially ordered topological vector spaces*, Clarendon Press, Oxford, 1973.
22. C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing, River Edge, NJ, 2002.

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